Wavelets transformation

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Deficiencies of Fourier transform

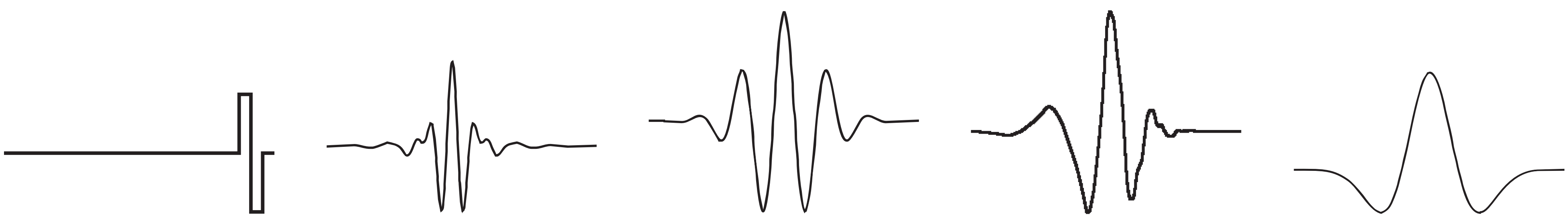
- Fourier transform and similar ones have a principal disadvantage: only the information about the frequency spectrum is provided, and no information is available on the *time* in 1D (or *location in the image* in 2D) at which events occur.
- One solution to the problem of localizing changes in the signal (image) is to use the short time Fourier transform, where the signal is divided into small windows and treated locally as it were periodic.
- The uncertainty principle provides guidance on how to select the windows to minimize negative effects, i.e., windows have to join neighboring windows smoothly.
- The window dilemma remains—a narrow window yields poor frequency resolution, while a wide window provides poor localization.

A more complex basis functions – wavelets

- The wavelet transform goes further than the short time Fourier transform.
- It also analyzes the signal (image) by multiplying it by a window function and performing an orthogonal expansion, analogously to other linear integral transformations.
- Formally, a wavelet series represents a square-integrable function with respect a complete, orthonormal set of basis functions called wavelets, meaning a small wave.
- There are two directions in which the analysis is extended with respect to Fourier transformation.
  1. The used basis functions (wavelets) are more complicated than sines and cosines applied in Fourier transform.
  2. The analysis is performed at multiple scales.

Wavelets

- Wavelets provide a localization in time (space) to a certain degree.
- The entire space-frequency localization is still not possible due to the Werner Heisenberg's uncertainty principle.
- In 1D, the shape of five commonly used basis functions in a single scale of many scales (mother wavelets) is illustrated pictorially in a qualitative manner;



Haar Meyer Morlet Daubechies-4 Mexican hat

Multiple scales

- Modeling a spike in a function (a noise dot in an image, for example) with a sum of a huge number of functions will be hard because of the spike strict locality.
- Functions that are already local will be naturally suited to the task.
- Such functions lend themselves to more compact representation via wavelets. Sharp spikes and discontinuities normally take fewer wavelet bases to represent as compared to the sine-cosine basis functions in Fourier transform.
- Localization in the spatial domain together with the wavelet localization in frequency yields a sparse representation of many practical signals (images).

- Sparseness opens the door to successful applications in data/image compression, noise filtering, detecting feature points in images, etc.

Parent and daughter wavelets

Mother wavelet Ψ, a wavelet function

- Characterizes the basic wavelet shape.
- Covers the entire domain of interest.

Father wavelet Φ, a scaling function

- Characterizes the basic wavelet scale.
- Allow to express needed details of the approximated function in the domain of interest.

All other derived wavelets are called daughter wavelets.

- Daughter wavelets are defined in terms of parent wavelets with the help of.
- the generating (basis) function $\Psi_{s,\tau}(x)$, where
  - s characterizes the scale of a wavelet function,
  - τ characterize shifts of a wavelet function.

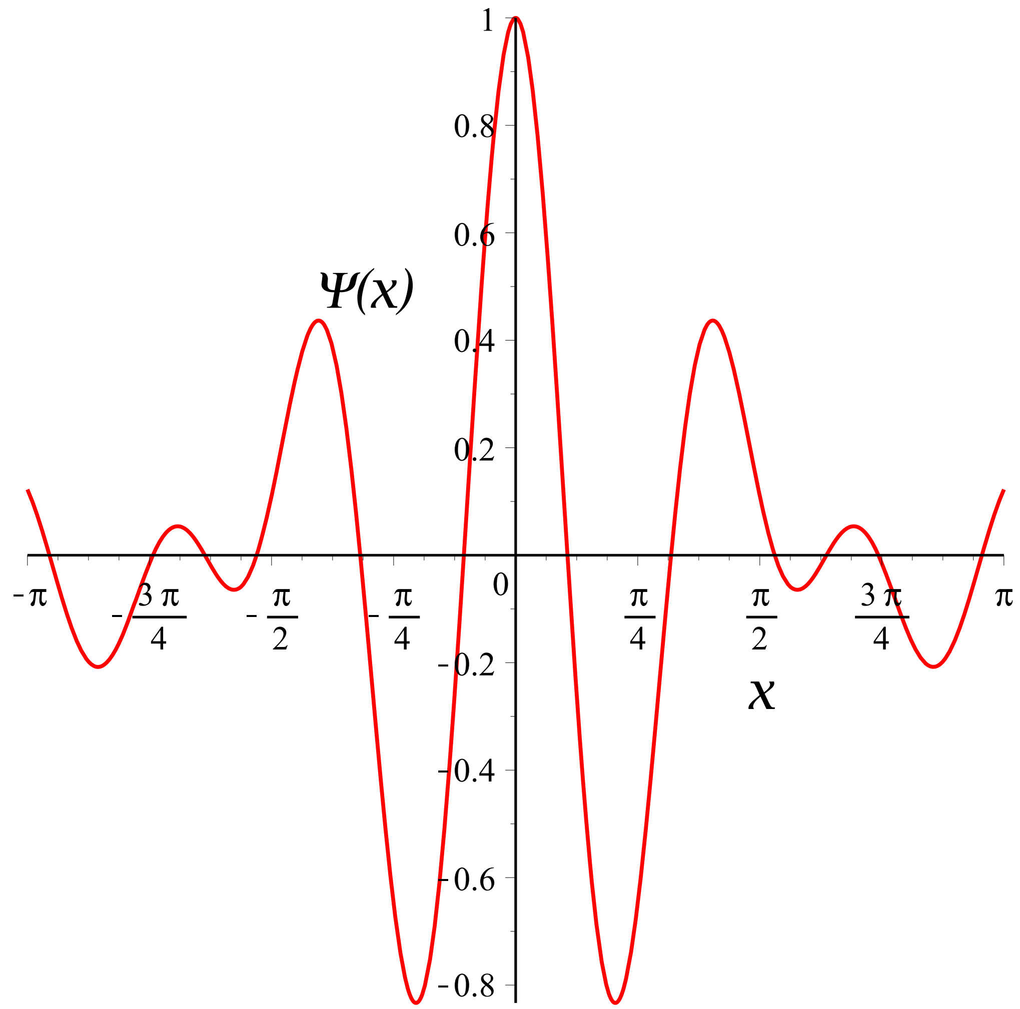
Continuous wavelet transforms (CWT)

- Continuous shift and scale parameters are considered.
- A given input signal of a finite energy is projected on a continuous family of frequency bands (subspaces of the function space L^p in functional analysis).
- For instance the signal may be represented on every frequency band of the form $[f, 2f]$ for all positive frequencies $f > 0$.
- The original signal can be reconstructed by a suitable integration over all the resulting frequency components.
- The frequency bands are scaled versions of a subspace at scale 1.
- This subspace is generated by the shifts of the mother wavelet Ψ.

The mother wavelet, the illustration example

For example, let us demonstrate the Shannon mother wavelet in one frequency band $[1, 2]$,

$$\Psi(t) = \frac{\sin(2\pi t) - \sin(\pi t)}{\pi t}\,.$$



1D continuous wavelet transform

- A function $f(t)$ is decomposed into a set of generating (basis) functions $\Psi_{s,\tau}(t)$, i.e. wavelets

$$c(s,\tau) = \int_R f(t)\, \Psi^*_{s,\tau}(t)\, \mathrm{d}t\,, \quad s \in R^+ - \{0\}\,, \quad \tau \in R\,.$$

  $c(s,\tau)$ are wavelet coefficients. The complex conjugation is denoted by *.

- The subscripts denote: s – scale, τ – shift (translation).
- Wavelets are generated from the single mother wavelet $\Psi(t)$ by scaling s and shifting τ; $s > 1$ dilates, $s < 1$ contracts the signal,

$$\Psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\, \Psi\left(\frac{t-\tau}{s}\right)\,.$$

- The coefficient $1/\sqrt{s}$ is used because the energy has to be normalized across different scales.

Meaning of wavelet coefficients $c(s, \tau)$

- The integral $\int_R f(t)\, \Psi^*_{s,\tau}(t)\, \mathrm{d}t$ from the previous slide can be interpreted as the scalar (inner) product of the signal $f(t)$ and the particular wavelet (basis function) $\Psi^*_{s,\tau}(t)$.
- This scalar product tells to what degree is the shape of the signal similar (correlated) to the local probe given by the particular wavelet.
- The space of scales s and shifts τ is discretized in real use. This discretization yields discrete wavelet transformation DWT. We will deal with the discretization later.

Inverse continuous wavelet transform

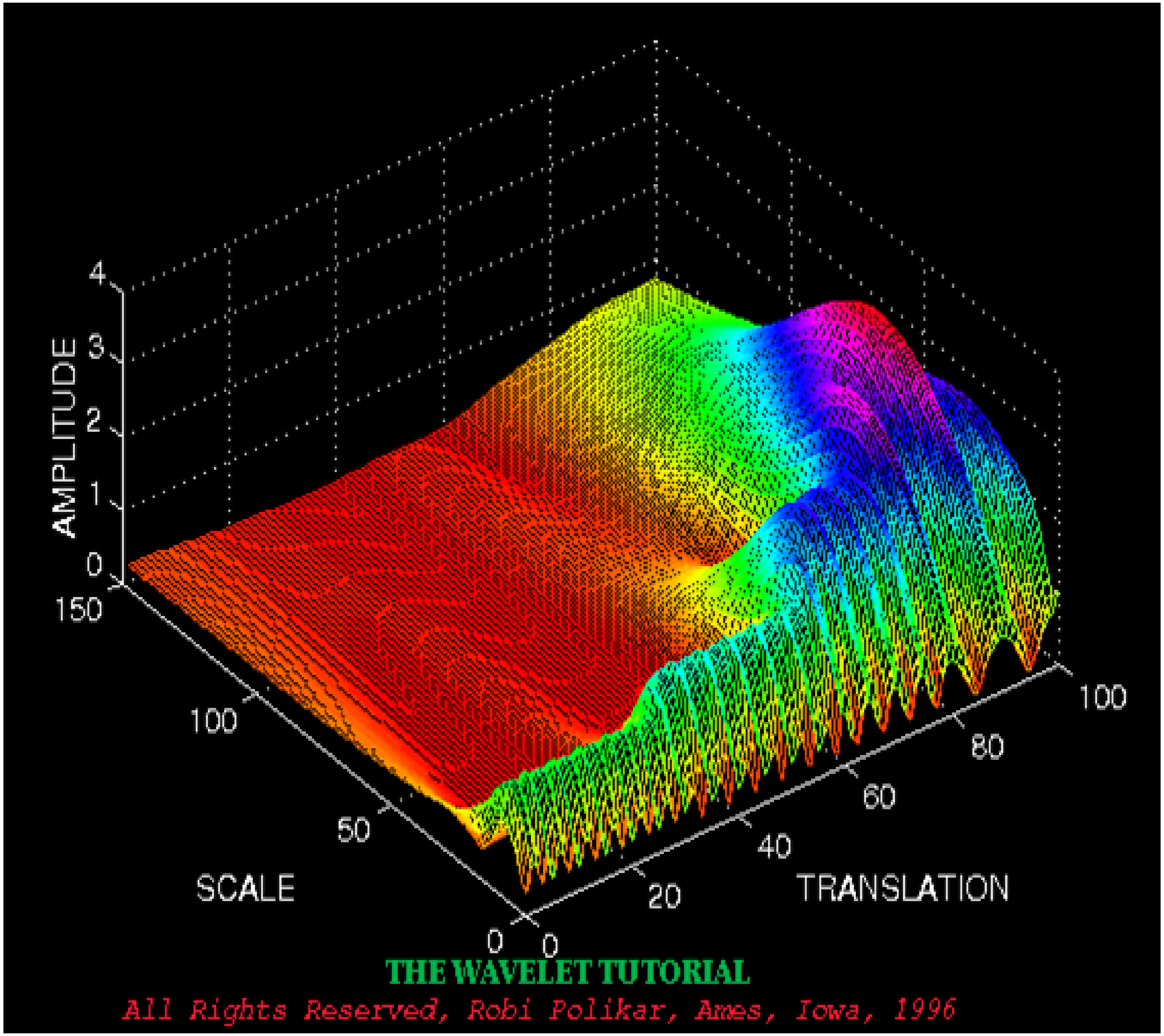
- The inverse continuous wavelet transform serves to synthesize the 1D signal $f(t)$ of finite energy from wavelet coefficients $c(s,\tau)$,

$$f(t) = \int_{R^+} \int_R c(s,\tau)\, \Psi_{s,\tau}(t)\, \mathrm{d}s\, \mathrm{d}\tau\ .$$

- Note:
  The wavelet transform was defined generally without the need to specify a particular mother wavelet Ψ. The user can select or design mother wavelet Ψ according to application needs. The mother wavelet is used to create generating (basis) functions $\Psi_{s,\tau}(t)$ used in the expansion above.

- Coefficients $c(s,\tau)$ can be interpreted as the analogy to a frequency spectrum (spectrogram) in Fourier transform. This is illustrated in the following transparency.

Wavelet “spectrogram” example



Q: Can any function be a mother wavelet? A: No.

- The mother wavelet should be oscillatory

$$\int_{-\infty}^{\infty} \Psi(t)\,\mathrm{d}\,t = 0\,.$$

- Mother wavelet has to have a finite energy

$$\int_{-\infty}^{\infty} |\Psi(t)|^2 \mathrm{d}\,t \leq \infty\,.$$

Wavelet vs. Fourier transform

- Wavelet transformation: spectral (‘frequency’) information and partly the information about the event in time (spatial coordinated in 2D).
- Fourier transformation: spectral (frequency) information only.

The ‘richness’ of the wavelet transformation with respect to Fourier transformation is not for free.

Wavelets:

- are not smooth, i.e. infinitely differentiable;
- loose spectral accuracy when computing derivatives;
- loose useful mathematical properties of Fourier transformation as, e.g., the convolution theorem.

Dyadic (octave) grid for scale and shift (1)

- The continuous change of scale s and shift τ parameters would lead to a very redundant signal representation.
- It is convenient to change scale and shift parameters in discrete steps.
- This is step towards the discrete wavelet transformation (DWT).

Dyadic (octave) grid for scale and shift (2)

- It is advantageous to use special values for shift τ and scale s while defining the wavelet basis, i.e. introducing the scale step j and the shift step k: $s = 2^{-j}$ and $\tau = k \cdot 2^{-j}$; $j = 1, \ldots$; $k = 1, \ldots$;

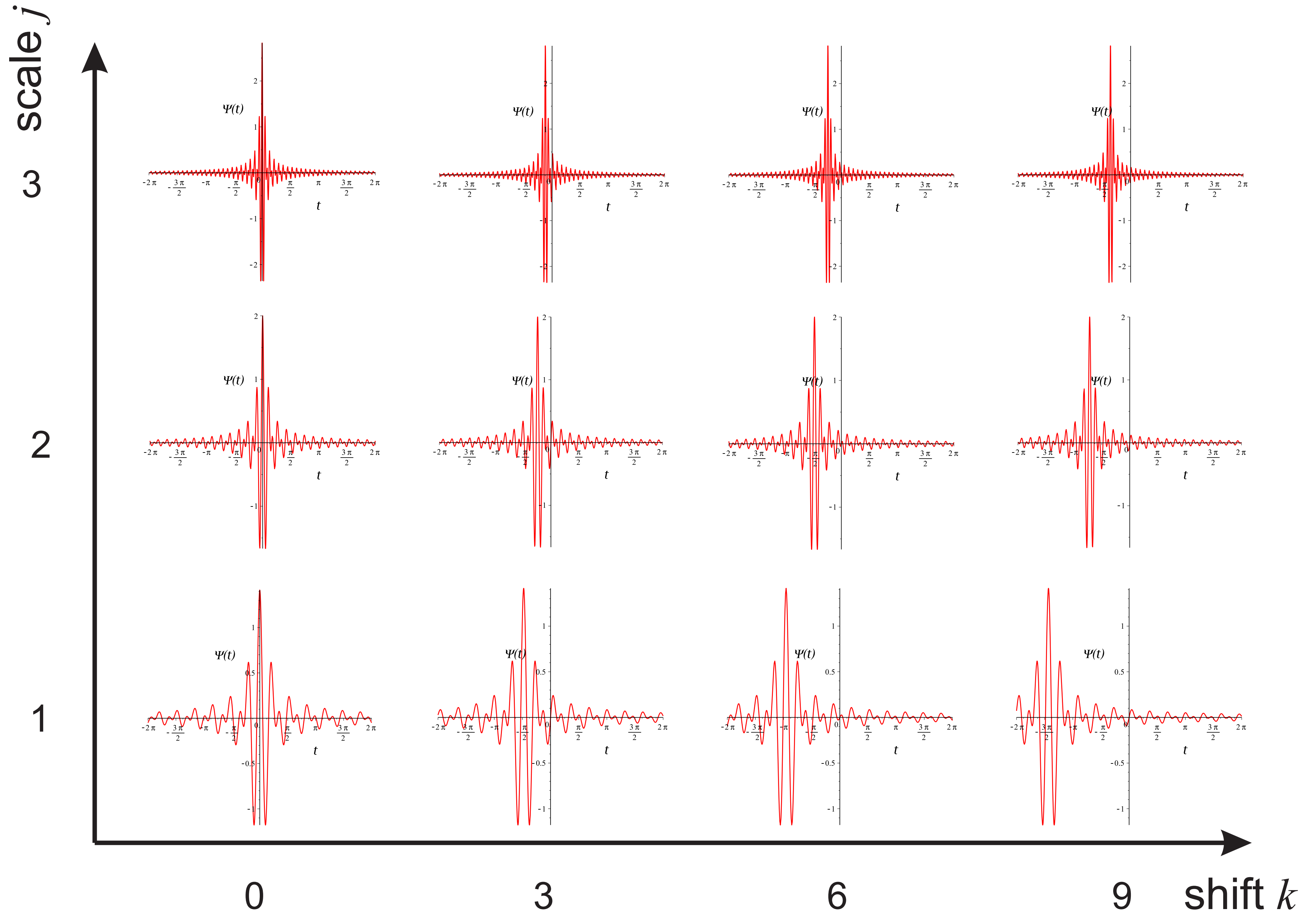
$$\Psi_{s,\tau}(t) = \frac{1}{\sqrt{s}} \Psi\left(\frac{t-\tau}{s}\right) = \frac{1}{\sqrt{2^{-j}}} \Psi\left(\frac{t - k\,2^{-j}}{2^{-j}}\right) = 2^{\frac{j}{2}}\,\Psi\left(2^j\,t - k\right) .$$

$$\Psi_{j,k}(t) = 2^{\frac{j}{2}}\,\Psi\left(2^j\,t - k\right) .$$

- Example (Shannon wavelet) expanded from the slide on the page 8:

$$\Psi_{j,k}(t) = 2^{\frac{j}{2}}\,\frac{(\sin(2\pi(2^j\,t + k)) - \sin(\pi\,2^j\,t + k))}{\pi\,(2^j\,t + k)} .$$

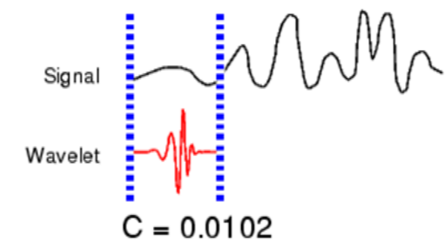
Example, Shannon wavelet, multiple scales, shifts



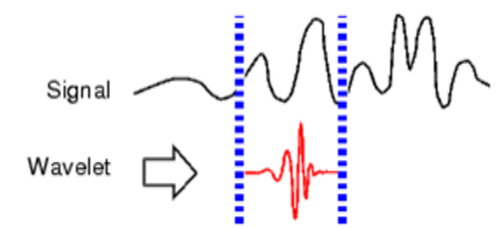
Computation of the Discrete Wavelet Transformation

1. Begins at a finest scale and a zero shift.
2. The wavelet is placed at beginning of the signal, the inner product of the signal and the wavelet is calculated, and integrated for all times.

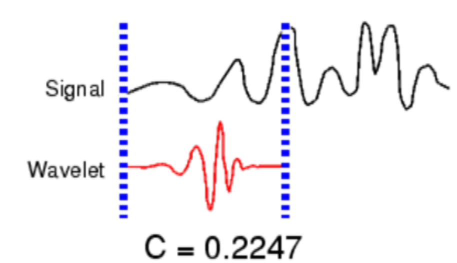
   The result is one value of $c(j,k)$ providing the 'local similarity' of a part of a signal and the wavelet.
3. The wavelet is shifted to the right and the step 2 is repeated until end of the signal.
4. The courser scale is used. Steps 2-3 repeated until all scales are used.



Finest scale, zero shift.



One of the shifts at finest scale.



Repeated for coarser scales.

The output is a matrix of c values for all scales and shifts, so called spectrogram.

Wavelets properties from a user point of view (1)

Simultaneous localization in time and in the 'frequency' spectrogram.

- The location of the wavelet allows to explicitly represent the location of events in time (with a theoretical limit given by Werner Heisenberg's uncertainty principle).
- The shape of the wavelet allows to represent different detail or resolution.

Sparsity of the representation – for practical signals: Many of the coefficients $c(j,k)$ in a wavelet representation are either zero or very small.

Linear computational time complexity – many 1D wavelet transformations can be accomplished in $\mathcal{O}(\mathcal{N})$ time.

Wavelets properties from a user point of view (2)

Adaptability – wavelets can be adapted to represent a wide variety of signals, e.g., functions with discontinuities, functions defined on bounded domains.

- Suited, e.g., for tasks involving closed or open curves, images, and very different surfaces in 3D representation.
- Wavelets can represent functions with discontinuities or corners (in images) rather efficiently. Recall that some wavelets have discontinuities themselves (or sharp corners in 2D case).

Discrete wavelet transformation (DWT)

- Uses discrete dyadic (octave) grid for scale parameter j and shift parameter k as introduced on the slide number 16.
- Forward DWT:

$$c(j,k) = \sum_t f(t)\,\Psi^*_{j,k}(t)\ , \ \text{where } \Psi_{j,k}(t) = 2^{\frac{j}{2}}\,\Psi\left(2^j\,t - k\right)\ .$$

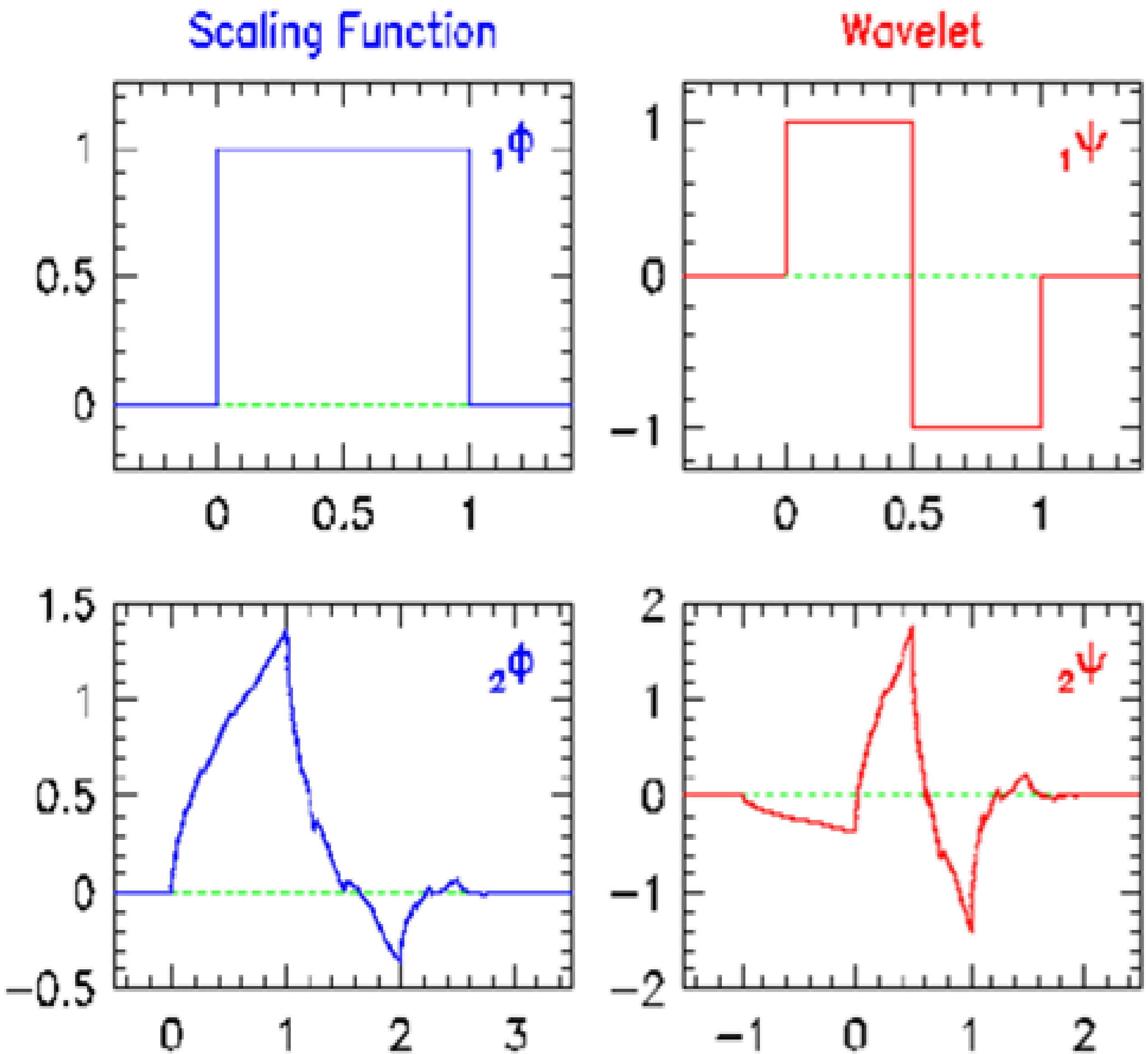
as was introduced on the slide number 16.

- Inverse DWT:

$$f(t) = \sum_k \sum_j c(j,k)\,\Psi_{j,k}(t)\ .$$

A. Haar and I. Daubechies wavelets
Pictorial example

- Haar wavelet
- Daubechies wavelet



Properties of Ingrid Daubechies' wavelets

Ingrid Daubechies, Communications Pure Applied Math. 41 (1988), 909-996.

- Compact support.
  - Finite number of filter parameters / fast implementations.
  - High compressibility.
  - Fine scale amplitudes are very small in regions where the function is smooth / sensitive recognition of structures.
- Identical forward / backward filter parameters.
  - Fast, exact reconstruction.
  - Very asymmetric.

Mallat's filter scheme
Fast Wavelet Transform

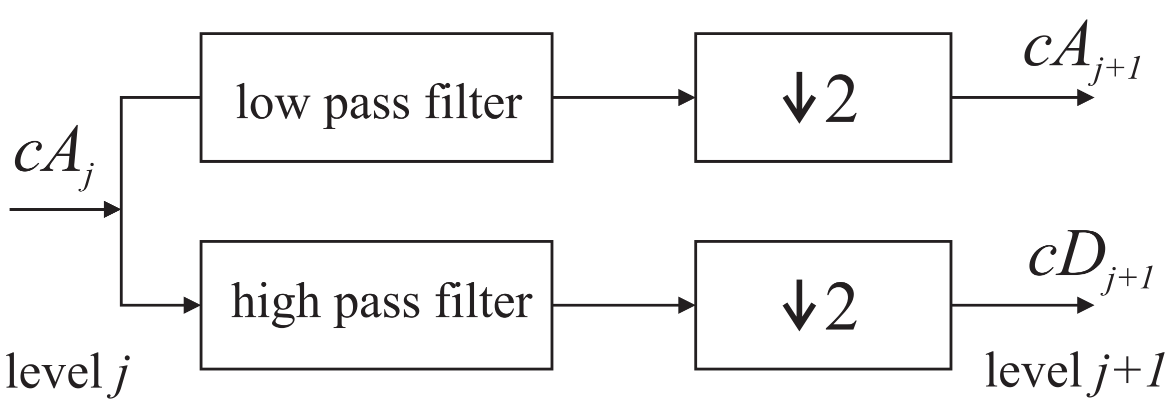
- Stephane G. Mallat: A Theory for Multiresolution Signal Decomposition: The Wavelet Representation, IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 11, No. 7, July 1989, pp. 674-693.
- S.G. Mallat was the first who implemented the dyadic grid scheme for wavelets using a well known filter design method called 'two channel sub band coder'.
- This yielded a 'Fast Wavelet Transform'.

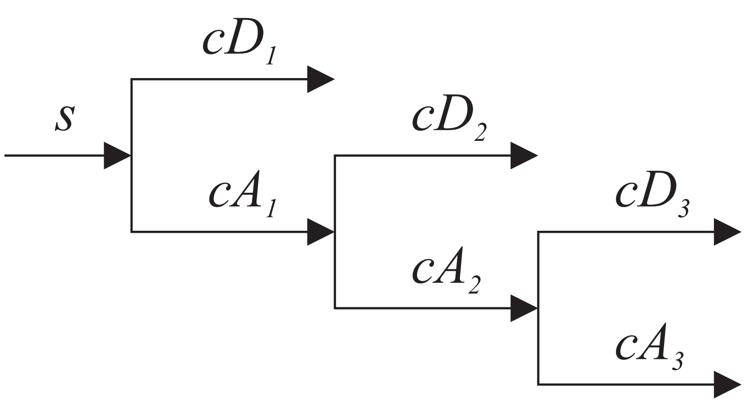
Fast Wavelet Transform

- Consider a discrete 1D signal given by the sequence s of length N which has to be decomposed into wavelet coefficients c.
- The Fast Wavelet Transform consists of $\log_2 N$ steps at most.
- The first decomposition step takes the input and provides two sets of coefficients at level 1: approximation coefficients cA_1 and detail coefficients cD_1.
- The vector s is convolved with a low-pass filter for approximation and with a high-pass filter for detail.
- Dyadic decimation follows which down samples the vector by keeping only its even elements. Such down sampling will be denoted by $\downarrow 2$ in block diagrams.

Fast Wavelet Transform, filter bank

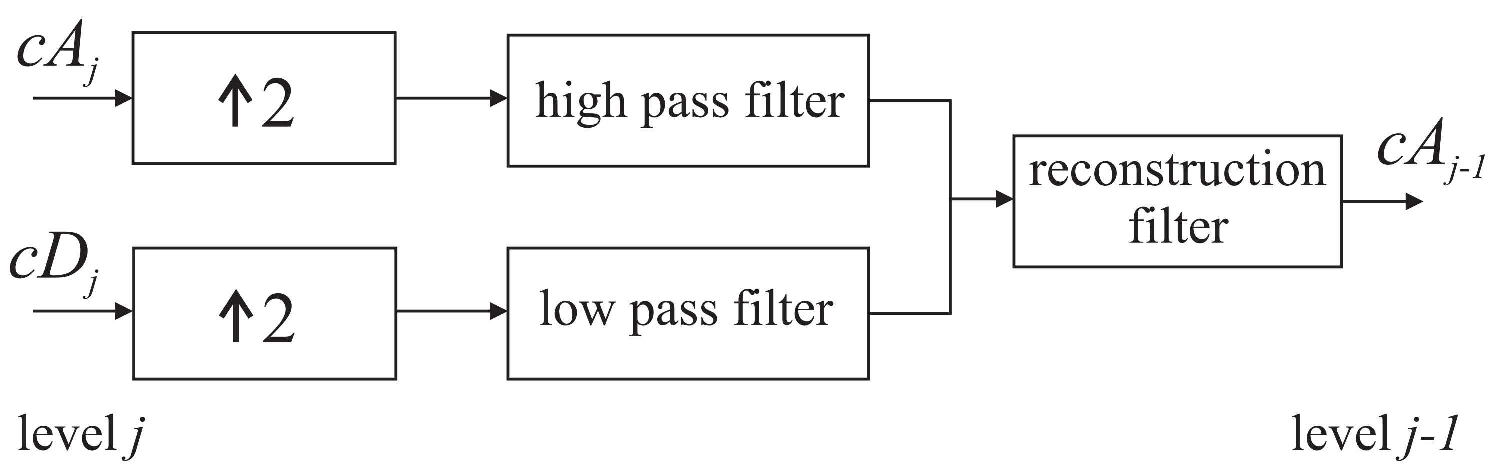
- The coefficients at level $j+1$ are calculated from the coefficients at level j, which is illustrated in the bottom-left figure.
- This procedure is repeated recursively to obtain approximation and detail coefficients at further levels. This yields a tree-like structure of filters called filter bank.
- The structure of coefficients for level $j=3$ is illustrated in the bottom-right figure.





Fast inverse Wavelet Transformation

- The Fast Inverse Wavelet Transform takes as an input the approximation coefficients cA_j and detail coefficients cD_j and inverts the decomposition step.
- The vectors are extended (up sampled) to double length by inserting zeros at odd-indexed elements and convolving the result with the reconstruction filters. Analogously to down sampling, up sampling is denoted $\uparrow 2$ in the block diagrams.



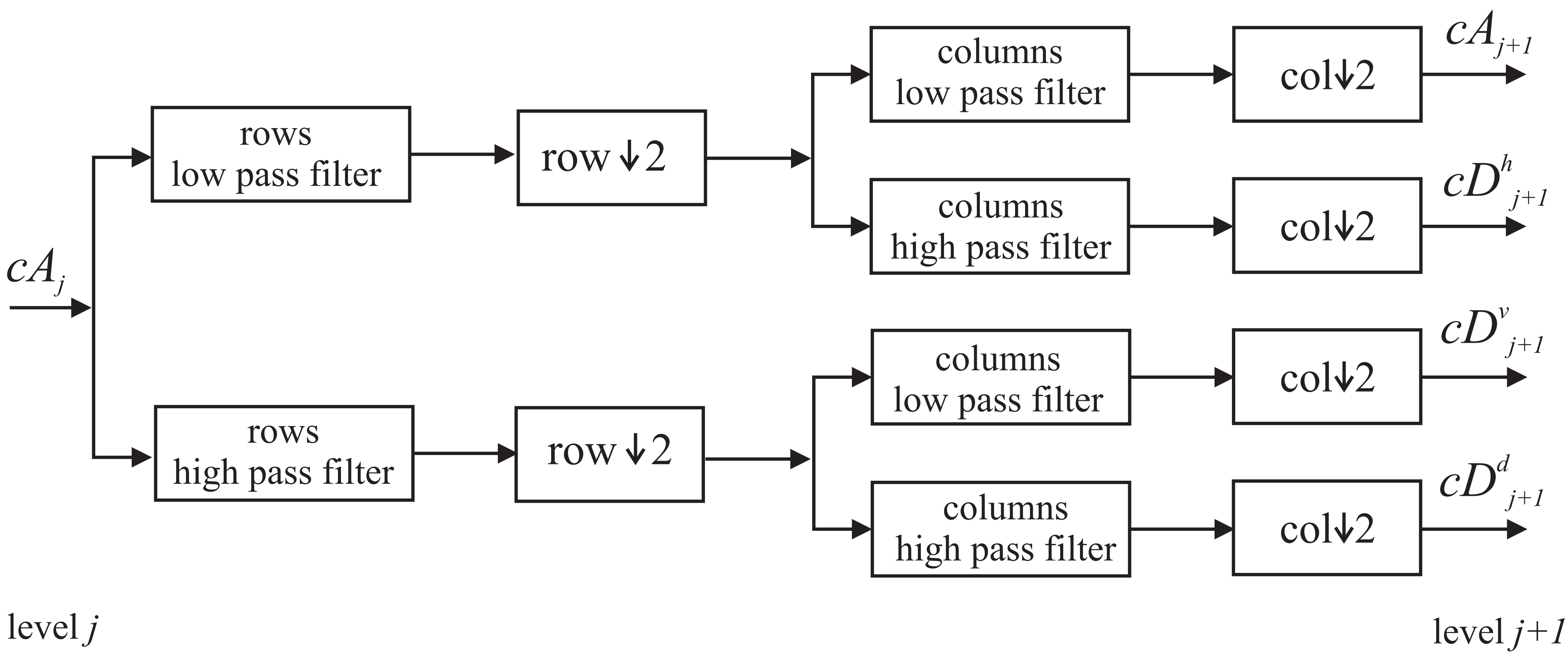
Wavelets generalized to 2D

Similar wavelet decomposition and reconstruction algorithms were developed for 2D signals (images). The 2D discrete wavelet transformation decomposes a single approximation coefficient at level j into four components at level $j+1$:

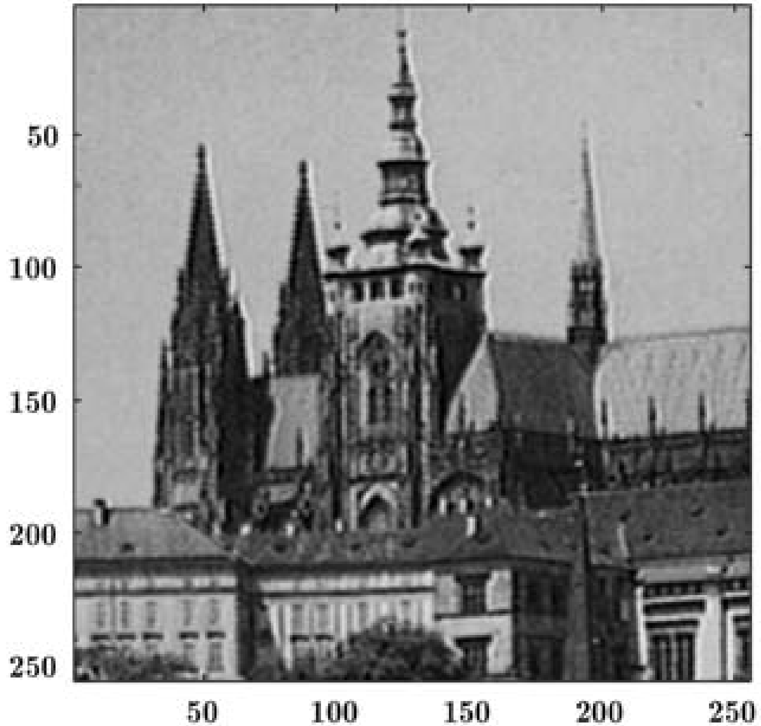
1. the approximation coefficient cA_{j+1} and detail coefficients at three orientations:
2. horizontal cD^h_{j+1} ,
3. vertical cD^v_{j+1} ,
4. and diagonal cD^d_{j+1} .

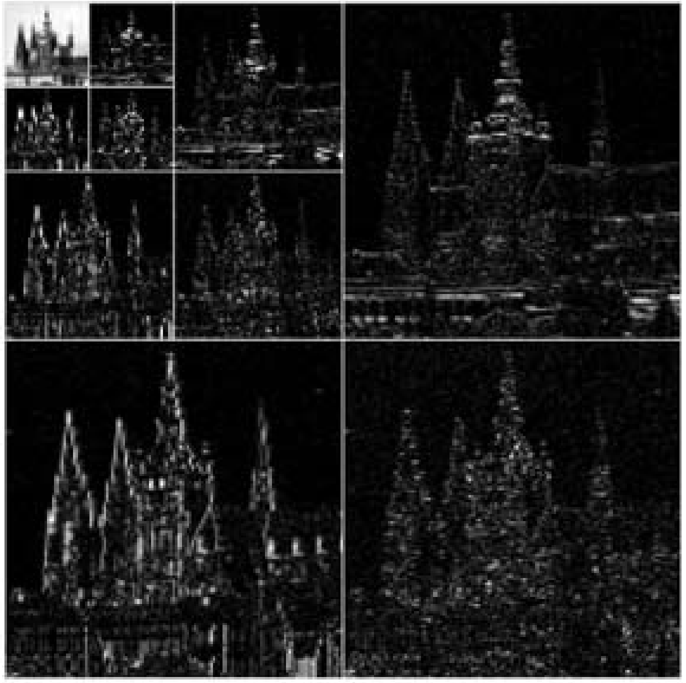
The symbol $(\mathrm{col} \downarrow 2)$ represents down-sampling columns by keeping only even indexed columns. Similarly, $(\mathrm{row} \downarrow 2)$ means down-sampling rows by keeping only evenly indexed rows. $(\mathrm{col} \uparrow 2)$ represents up-sampling columns by inserting zeros at odd-indexed columns. Similarly, $(\mathrm{row} \uparrow 2)$ means up-sampling rows by inserting zeros at odd-indexed rows.

2D discrete wavelet transform
A decomposition step



Example, 2D wavelet decomposition

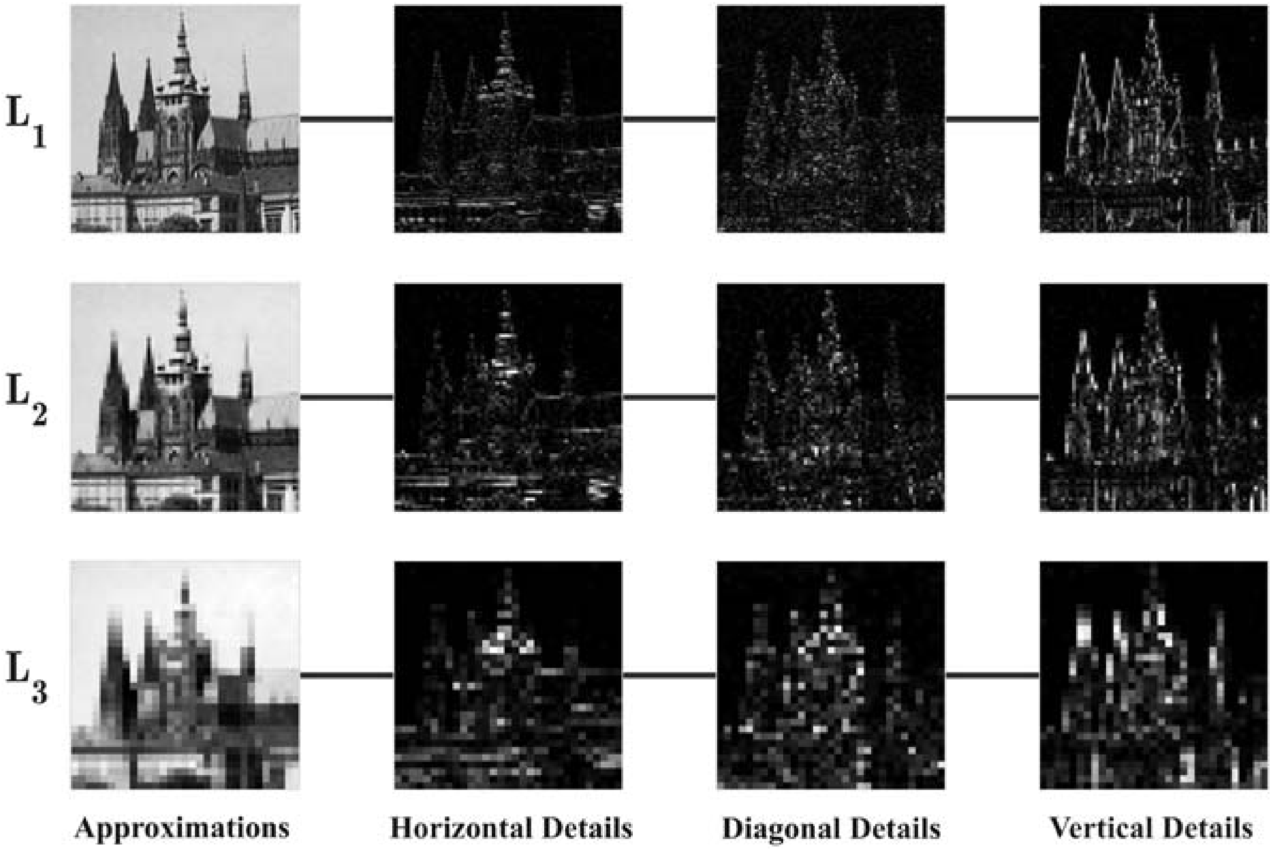




Decomposition at level 3

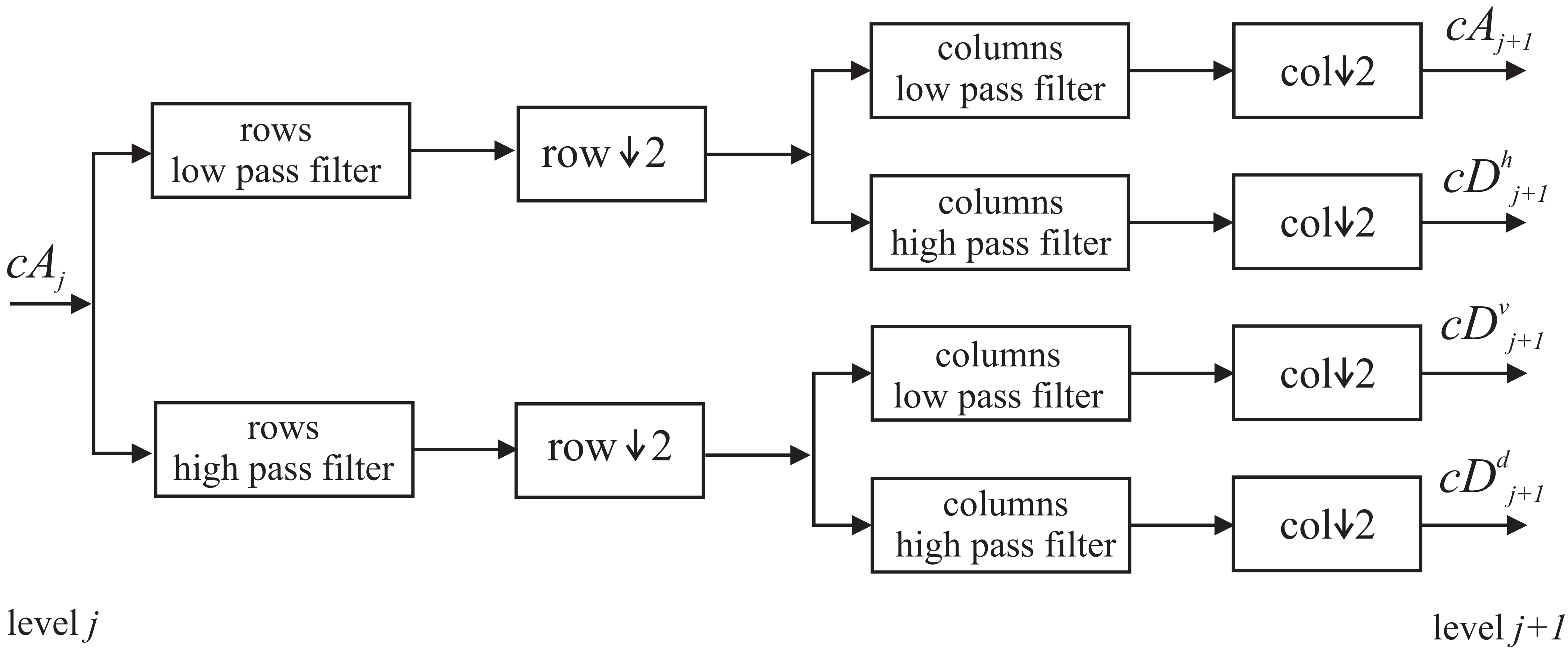
Right side. Four quadrants. The undivided southwestern, southeastern and northeastern quadrants correspond to detailed coefficients of level 1 at resolution 128×128 in vertical, diagonal and horizontal directions, respectively. The northwestern quadrant displays the same structure for level 2 at resolution 64×64. The northwestern quadrant of level 2 shows the same structure at level 3 at resolution 32×32. The lighter intensity 32×32 image at top left corresponds to approximation coefficients at level 3.

Example, 2D wavelet decomposition, another view



Another view of the same data as on previous slide. It illustrates the level of decomposition at different levels of resolution.

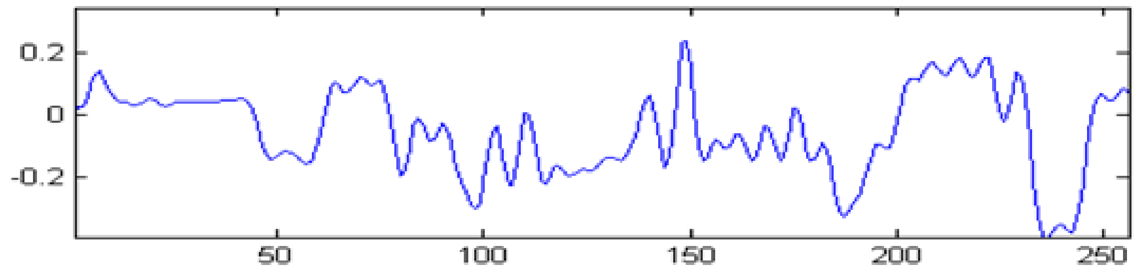
2D inverse discrete wavelet transform. A reconstruction step.



Use of filter bank, illustration

The lower resolution coefficients can be calculated from the higher resolution coefficients by a tree-structured algorithm, the filter bank.

$$f_1(t) = \sum_k \sum_j c(1,k)\ \Psi_{1,k}(t)$$



$$f_0(t) = \sum_k \sum_j c(0,k)\ \Psi_{0,k}(t)$$