Mathematical morphology

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Outline of the talk:

- Point sets. Morphological transformation.
- Erosion, dilation, properties.
- Opening, closing, hit or miss.
- Skeleton.
- Thinning, sequential thinning.
- Distance transformation.
Morphology is a general concept

**In biology:** the analysis of size, shape, inner structure (and relationship among them) of animals, plants and microorganisms.

**In linguistics:** analysis of the inner structure of word forms.

**In materials science:** the study of shape, size, texture and thermodynamically distinct phases of physical objects.

**In signal/image processing:** mathematical morphology – a theoretical model based on lattice theory used for signal/image preprocessing, segmentation, etc.
Mathematical morphology, introduction

Mathematical morphology (MM):

- is a theory for analysis of planar and spatial structures;
- is suitable for analyzing the shape of objects;
- is based on a set theory, integral algebra and lattice algebra;
- is successful due to a simple mathematical formalism, which opens a path to powerful image analysis tools.

The morphological way . . .

The key idea of morphological analysis is extracting knowledge from the relation of an image and a simple, small probe (called the structuring element), which is a predefined shape. It is checked in each pixel, how does this shape matches or misses local shapes in the image.
Mathematical morphology pioneers

Georges Matheron
*1930, †2000

Jean Serra
*1940

Ivan Saxl
*1936, †2009

Josef Mikeš
*1946


The first promoters of the mathematical morphology in Czechoslovakia were Ivan Saxl and Josef Mikeš in 1970s.
Additional literature

- Course on mathematical morphology by Jean Serra: http://www.cmm.mines-paristech.fr/~serra/cours/
- Laurent Najman and Hugues Talbot (editors), Mathematical Morphology, John Wiley & Sons, Inc., London, 2010
**Links with other theories and approaches**

Mathematical morphology does not compete with other theories, it complements them.

- Discrete geometry (e.g., distance, skeleton of a region).
- Graph theory, e.g., minimal spanning tree (O. Boruvka 1926), watershed, computational geometry.
- Statistics: random models, measure theory, stereology, etc.
- Linear signal theory: replace $+$ with the supremum $\lor$.
- Scale-space: replace Gaussian smoothing with openings/closings $\Rightarrow$ granulometry.
- Level sets: dilations with PDEs, FMM (Fast Marching Method) is a distance function.
- *Note: there are no equivalents of Fourier and wavelet transformations in mathematical morphology.*
Different mathematical structures used

Signal processing in a vector space

Vector space is a set of vectors $V$ and set of scalars $K$, such that

- $K$ is a field.
- $V$ is a commutative group.
- Vectors can be added together and multiplied (scaled) by scalars.

Mathematical morphology

The complete lattice $(E, \sqsubseteq)$ is a set $E$ provided with an ordering relation $\sqsubseteq$, such that

- $\forall x, y, z \in E$ holds (partial ordering)
  
  $x \sqsubseteq x,$
  
  $x \sqsubseteq y, y \sqsubseteq x \Rightarrow x = y,$
  
  $x \sqsubseteq y, y \sqsubseteq z \Rightarrow x \sqsubseteq z.$

- For any $P \subseteq E$ there exists in $E$ (completeness)
  
  - A greatest lower bound $\bigwedge P$, called infimum (also called meet).
  
  - A lowest upper bound $\bigvee P$, called supremum (also called join).
Example of Lattices

- Lattice of primary additive colors (RGB).
- Lattice of real numbers $\mathbb{R}$.
- Lattice of real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.
- Lattice of whole numbers (integers) $\overline{\mathbb{N}} = \mathbb{N} \cup \{-\infty, +\infty\}$.
- The Cartesian product of the natural numbers, ordered by $\leq$ so that $(a, b) \leq (c, d) \iff (a \leq c) \& (b \leq d)$. 
Examples of complete lattices useful in image analysis

- Boolean lattice of sets ordered by inclusion $\Rightarrow$ binary mathematical morphology, where, e.g., the occupancy of a pixel is of interest.

- Lattice of upper semicontinuous functions $\Rightarrow$ gray level mathematical morphology or binary mathematical morphology in 3D binary images, where the occupancy of a voxel is of interest.

  *Note: Generalization to higher dimensions is possible, e.g. for $n$-dimensional images, as well as for multi-valued functions, e.g. time series as in motion analysis.*

- Lattice of multi-valued functions $\Rightarrow$ mathematical morphology for color images.
Comparison of basic operations

Linear signal processing

- Based on the ‘superposition principle’, fundamental laws are addition, multiplication and scalar product.

- Basic operations preserve addition and multiplication and commute under them.

\[ \Psi \left( \sum_i \lambda_i f_i \right) = \sum_i \lambda_i \Psi(f_i). \]

- The important operation is called convolution. It allows finding relation between two functions.

Mathematical morphology

- The lattice is based on the ordering, supremum \( \lor \) and infimum \( \land \). Basic operations preserve the supremum and the infimum.

- Ordering preservation

\[ \{ x \sqsubseteq y \Rightarrow \Psi(x) \sqsubseteq \Psi(y) \} \iff \] the operation \( \Psi \) is increasing.

- Commutation under the supremum

\[ \Psi(\lor x_i) = \lor \Psi(x_i) \iff \text{dilation.} \]

- Commutation under the infimum

\[ \Psi(\land x_i) = \land \Psi(x_i) \iff \text{erosion.} \]
Symmetry of supremum and infimum

- The supremum $\lor$ and the infimum $\land$ play a symmetrical role in a lattice.
  - They are exchanged if we exchange ordering $x \sqsubseteq y \iff x \sqsupseteq y$.
  - This leads to the concept of duality.

- Example: In a lattice of all subsets of a set $E(2^E; \subseteq)$, two operations $\Psi$ and $\Psi^*$ are called dual, iff

$$\Psi(X^C) = [\Psi^*(X)]^C,$$

where $X^C = E \setminus X$ denotes a complement of $X$ in $E$. 
The operation $\Psi$ is **extensive** on a lattice $(E, \sqsubseteq)$ iff, for all elements in $E$, the transformed element is greater than or equal to the original element, i.e.

$$\Psi \text{ is extensive} \iff \forall x \in E, \quad x \sqsubseteq \Psi(x).$$

The operation $\Psi$ is **anti-extensive** on a lattice $(E, \sqsubseteq)$ iff, for all elements in $E$, the transformed element is lower than or equal to the original element, i.e.

$$\Psi \text{ is antieextensive} \iff \forall x \in E, \quad \Psi(x) \sqsubseteq x.$$
Why so abstract?

- We can define operations that work in a general way.
- The operations can be studied without specifying the definition space.
- As a result, operations can be applied to, e.g., discrete images, continuous images, graphs, or meshes.
- Q: How the lattice framework can be used for images?
  A: Images are often considered as a function $f : E \to T$, where $E$ is the set of image points (pixels) and $T$ is a set of possible pixel values (more later).
Lattices of functions

- Let \( E \) be arbitrary set and \( T \) a closed subset of \( \overline{\mathbb{R}} \) or \( \overline{\mathbb{Z}} \).

- Functions \( f : E \rightarrow T \) generate a new lattice denoted \( T^E \) (the abbreviation for the Cartesian product \( T \times T \times \ldots \times T \), \( |E| \) times),

\[
    f \sqsubseteq g, \text{ iff } f(x) \leq g(x) \text{ for } \forall x \in E,
\]

where the supremum and the infimum derive directly from those of \( T \),

\[
    \left( \bigvee f_i \right)(x) = \bigvee f_i(x) \quad \left( \bigwedge f_i \right)(x) = \bigwedge f_i(x).
\]

- The approach extends directly to multivariate functions (e.g., color images, motion).
A simple case, the lattice for binary images

- There are several ways how to define a lattice and induced morphological operation.

- Let us start from a simple, intuitive and practically useful case of binary images

\[ f : E \to \{0, 1\} , \]
\[ F = \{ x \in E \mid f(x) = 1 \} . \]

- The lattice structure \((E, \leq)\) for binary images can be introduced as:

\[ (2^E, \subseteq) , \]

where \( X \subseteq Y \Leftrightarrow X \subseteq Y , \wedge \leftrightarrow \cap, \vee \leftrightarrow \cup. \]
Point sets

- Images can be represented by point sets of an arbitrary dimension, e.g. in a $N$-dimensional Euclidean space.

- 2D Euclidean space $\mathbb{E}^2$ and a system of its subsets is a natural continuous domain representing planar objects.

- A digital counterpart of the Euclidean space based on integer numbers $\mathbb{Z}$.

- **Binary mathematical morphology in 2D** – a set of points represented as pairs of integers $((x, y) \in \mathbb{Z}^2)$. The presence of the point informs about occupancy of a particular pixel (a location in a lattice).

- **Binary mathematical morphology in 3D** – a set of points represented as triplets of integers $(x, y, z) \in \mathbb{Z}^3$, where $(x, y, z)$ are volumetric coordinates informing about occupancy of a particular voxel.

- **Grayscale mathematical morphology in 2D** – a set of points $(x, y, g) \in \mathbb{Z}^3$, where $x, y$ are coordinates in a plane and $g$ represents the gray value of a particular pixel.
Four principles of mathematical morphology

1. **Compatibility with translation** – The morphological operator $\Psi$ should not depend on the translation.

2. **Compatibility under the scale change** – The morphological operator $\Psi$ should not depend on the scale.
   
   *Note: This principle is necessarily (slightly) violated for digital images.*

3. **Local knowledge** – The morphological operator $\Psi$ is a local operator (see structuring elements soon).

4. **Semi-continuity** – The morphological operator should not exhibit abrupt changes of its behavior.
A structuring element serves as a local probe in morphological operators.

Structuring elements are expressed with respect to local coordinates with the origin in the representative point $\mathcal{O}$ (denoted by $\times$ in subsequent figures).

Examples:
We start with the binary morphology, point set

- We will constrain for a while to binary mathematical morphology.
- The example of a point set in $\mathbb{Z}^2$,

\[ X = \{(1, 0), (1, 1), (1, 2), (2, 2), (0, 3), (0, 4)\} \]
Translation $X_h$ of a point set $X$ by a radiusvector $h$

$$X_h = \{ p \in \mathbb{E}^2, \ p = x + h \text{ for some } x \in X \}.$$
Symmetric point set

- Central symmetry is expressed with respect to a representative point \( O \).
- The alternative name is a transposed point set.
- Definition: \( \tilde{B} = \{-b : b \in B\} \).
- Example: \( B = \{(2, 1), (2, 2)\} \), \( \tilde{B} = \{(-2, -1), (-2, -2)\} \).
Deals with binary images. The domain is $\mathbb{Z}^2$. The range is $\{0, 1\}$.

Two basic operations: dilatation and erosion, which are not invertible but dual.

Two used formalisms for addition and subtraction

- The ordinary addition and subtraction as taught in the elementary schools.
- The difference between these two approaches plays a role with erosion.
Minkowski addition, subtraction

**Minkowski addition** (Hermann Minkowski 1864-1909, Geometry of numbers 1889)

\[ X \oplus B = \bigcup_{b \in B} X_b . \]

**Minkowski subtraction** (the concept was introduced by H. Hadwiger in 1957)

\[ X \ominus B = \bigcap_{b \in B} X_{-b} . \]
Binary dilation $\oplus$

- The dilation is a Minkowski addition, i.e. the union of translated point sets,

\[ X \oplus B = \bigcup_{b \in B} X_b. \]

- Dilation operation $X \oplus B$ was denoted $\delta_B(X)$ in a functional form by J. Serra.

- The dilation operation $\oplus$ can be equivalently expressed as the function $\delta$,

\[ \delta_B(X) = X \oplus B = \{ p \in \mathbb{R}^2 : p = x + b, \ x \in X \text{ and } b \in B \}. \]
Binary dilation $\oplus$, an example

\[ X = \{(1,0), (1,1), (1,2), (2,2), (0,3), (0,4)\} \]
\[ B = \{(0,0), (1,0)\} \]
\[ X \oplus B = \{(1,0), (1,1), (1,2), (2,2), (0,3), (0,4), (2,0), (2,1), (2,2), (3,2), (1,3), (1,4)\} \]
Binary dilation
with an isotropic structuring element $3 \times 3$

left – original, right – dilation.

Dilation is used to fill small holes and narrow bays in objects. The size of the object increases. If the size has to be kept similar then dilation is combined with erosion (to come).
**Dilation properties**

**Commutative:** \( X \oplus B = B \oplus X \).

**Associative:** \( X \oplus (B \oplus D) = (X \oplus B) \oplus D \).

**Invariant to translation:** \( X_h \oplus B = (X \oplus B)_h \).

**Increasing transformation:** If \( X \subseteq Y \) and \((0,0) \in B\) then \( X \oplus B \subseteq Y \oplus B \).

*Counterexample for the case of the empty representative point \((0,0) \notin B*
Binary erosion ⊖

- The erosion is a Minkowski subtraction, i.e. the intersection of all image $X$ translations by the vector $-b \in B$,

$$X \ominus B = \bigcap_{b \in B} X_{-b}.$$ 

- Erosion operation $X \ominus B$ was denoted $\varepsilon_B(X)$ in a functional form by J. Serra.

- Equivalently, it is verified for each image pixel $p$, whether the result fits to $X$ for all possible $x + b$. If yes then the outcome is 1, otherwise 0.

$$\varepsilon_B(X) = X \ominus B = \{p \in \mathbb{E}^2 : p = x + b \in X \text{ for all } b \in B\}.$$ 

- Dilation and erosion are dual morphological operations.
Binary erosion $\ominus$, an example

\[
X = \{(1,0), (1,1), (1,2), (0,3), (1,3), (2,3), (3,3), (1,4)\}
\]

\[
B = \{(0,0), (1,0)\}
\]

\[
X \ominus B = \{(0,3), (1,3), (2,3)\}
\]
Binary erosion with an isotropic structuring element $3 \times 3$

Objects smaller than the structuring element vanish (e.g. lines of length one).

Erosion is used for simplifying structure (decomposition of an object into simpler parts).
Contour of a binary region $\partial X$ (boundary mathematically) has a natural width one.

$$\partial X = X \setminus (X \ominus B).$$

left – original $X$, right boundary (contour) $\partial X$. 
Erosion properties

**Antiextensive:** If $(0, 0) \in B$ then $X \ominus B \subseteq X$.

**Invariant with respect to translation:** $X_h \ominus B = (X \ominus B)_h$, 
$X \ominus B_h = (X \ominus B)_{-h}$.

**Preserves inclusion:** If $X \subseteq Y$ then $X \ominus B \subseteq Y \ominus B$.

**Duality of erosion and dilation:** $(X \ominus Y)^C = X^C \ominus Y$

**Combination of erosion and intersection:** 
$(X \cap Y) \ominus B = (X \ominus B) \cap (Y \ominus B)$, 
$B \ominus (X \cap Y) \supseteq (B \ominus X) \cup (B \ominus Y)$. 
Dilation and erosion properties (2)

The order of dilation and erosion can be interchanged:

\[(X \cap Y) \ominus B = B \ominus (X \cap Y) \subseteq (X \oplus B) \cap (Y \oplus B).\]

Dilation of the intersection of two sets (images) is contained in the intersection of their dilations.

A possible interchange of erosion and set intersection (e.g. enables decomposition of more complex structural elements into simpler ones):

\[B \ominus (X \cup Y) = (X \cup Y) \ominus B = (X \oplus B) \cup (Y \ominus B),\]

\[(X \cup Y) \ominus B \supseteq (X \ominus B) \cup (Y \ominus B),\]

\[B \ominus (X \cup Y) = (X \ominus B) \cap (Y \ominus B).\]
Dilation and erosion properties (3)

Sequential dilation (resp. erosion) of the image \( X \) by the structural element \( B \) followed by the structural element \( D \) is equivalent to dilation (resp. erosion) of the image \( X \) by \( B \oplus D \)

\[
(X \oplus B) \oplus D = X \oplus (B \oplus D),
\]
\[
(X \ominus B) \ominus D = X \ominus (B \oplus D).
\]
Morphological filtering

- In ‘classical’ signal/image processing, the term filter denotes an arbitrary processing procedure having a signal/image both as an input and an output.

- A filter has a precise meaning in mathematical morphology, i.e.
  
  Operation $\Psi$ is a morphological filter $\iff$ $\Psi$ is increasing and idempotent.

- In words: morphological filters preserve ordering relation and converge in a finite number of iterations.

- Two most important operations are openings and closings in this context.

- Openings are anti-extensive morphological filters.

- Closings are extensive morphological filters.
Binary opening $\circ$

Erosion followed by dilation.

$$X \circ B = (X \ominus B) \oplus B$$

If the image $X$ remains unchanged after opening by the structuring element $B$, it is open with respect to $B$. 
Binary closing

Dilation followed by erosion.

\[ X \bullet B = (X \oplus B) \ominus B \]

If the image \( X \) remains unchanged after closing with the structural element \( B \), it is closed with respect to \( B \).
Properties of opening, closing

Opening and closing are dual morphological transformations

\[(X \bullet B)^C = X^C \circ \tilde{B}\]

Idempotence – is a property of algebraic operations or elements of the algebra. The operation is idempotent, if its repeated use on some input provides the same output, which is yielded by a single use of that operation.

Here specially: after one opening, resp. closing, the set is open, resp. closed. Subsequent use of these transformations does not change anything.

\[X \circ B = (X \circ B) \circ B\]

\[X \bullet B = (X \bullet B) \bullet B\]
Transformation hit or miss \(\otimes\)

- Uses a composite structuring element \(B = (B_1, B_2), B_1 \cap B_2 = \emptyset\).

\[
X \otimes B = \{x : B_1 \subset X \text{ and } B_2 \subset X^c\}.
\]

- Indicates the match between the composite structuring element and the part of the image. \(B_1\) checks the objects and \(B_2\) the background.

- Transformation \(\otimes\) can be expressed by erosions and dilations

\[
X \otimes B = (X \ominus B_1) \cap (X^c \ominus B_2) = (X \ominus B_1) \setminus (X \oplus \hat{B}_2).
\]
Example: detection of convex corners

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Masks detecting four possible configuration of convex corners using hit or miss transformation.

The result of corner detection.
Homotopic transformation

- It is based on the connectedness relation between points, regions, which are expressed by a homotopic tree.

- The homotopic transformation does not change the homotopic tree. The topology relations remain unchanged.

Example: The same homotopic tree corresponds to two different images.
It is natural to represent elongated objects by a skeleton.

H. Blum proposed in 1967 a “Medial axis transformation”. A locus of points equidistant from contour (the analogy: grass fire).

A formal definition of the skeleton is based on the concept of the maximal circle (the ball in 3D).
Skeleton by maximal balls

- A circle $B(p, r)$ with the center $p$ and the radius $r$, where $r \geq 0$ is a set of points, for which the distance $d \leq r$.

- The maximal circle $B$ inscribed into the set $X$ touches its boundary $\partial X$ in two or more points.

- The skeleton is a union of maximal circles centers.
Skeleton example, continuous case

Troubles with noise.
Discrete circles with the radius 1

Circles look differently in the discrete lattice due to different ways how the distance is defined.

Examples:
Taxonomy of binary skeletonization algorithms

- **Inscription of circles** based on previous definition is almost unused in practice. The computational complexity is high. The connectivity is being lost. The skeleton width $> 1$.

- **Sequential thinning.** The region is eroded by a suitable structuring element, which ensures that the connectivity is not broken. The oldest approach uses structuring elements from the Golay alphabet (1969) providing a homotopic thinning.

- **Via Voronoi diagram.** Computationally expensive process, especially for large and complex objects.

- **Via distance transformation.** Fast and most often used.

- **Via corner representation.** Regions are first compressed losslessly (providing corners). The skeleton is obtained by inscribing maximal rectangles directly in the compressed data (Schlesinger M.I., 1986).
Properties of discrete skeleton approximations

- Approximations of a true skeleton are available in discrete spaces (images).

- The skeleton approximation should comply to two requirements
  - Topology, i.e. contiguity has to be pertained, e.g. a homotopic tree.
  - Geometry, i.e. parts of the skeleton should be “in the middle” of the object and be invariant to geometric transformations as shift, rotation, scaling.

- Comparison

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<thead>
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<td>Via Voronoi diagram</td>
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<td>yes</td>
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<tr>
<td>Via distance transformation</td>
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<td>yes</td>
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<td>Via Schlesinger’s corners</td>
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**Thinning and thickening**

- Let $X$ be an image and $B = (B_1, B_2)$ be a **composite structuring element** introduced for the hit or miss transformation.

- **Thinning** $X \ominus B = X \setminus (X \otimes B)$.
  
  When thinning, a part of the boundary of the object given by the structuring element $B$ is subtracted from it by the set difference operation.

- **Thickening** $X \oplus B = X \cup (X^c \otimes B)$.
  
  When thickening, a part of the boundary given by the structuring element $B$ of the background is added to the object.

- Thinning and thickening are **dual transformations**
  
  $$(X \ominus B)^c = X^c \ominus B, \quad B = (B_2, B_1).$$
Sequential thinning and thickening

- Let \( \{B_1, B_2, B_3, \ldots, B_n\} \) be a sequence of composite structuring elements \( B_i = (B_{i1}, B_{i2}) \).

- **Sequential thinning** can then be expressed as a sequence of \( n \) structuring elements for square rasters (e.g. eight elements of the size \( 3 \times 3 \), as will be shown later with Golay alphabet).

\[
X \ominus \{B_i\} = (((X \ominus B_1) \ominus B_2) \ldots \ominus B_n) .
\]

- **Sequential thickening** (analogically)

\[
X \circ \{B_i\} = (((X \circ B_1) \circ B_2) \ldots \circ B_n) .
\]
Useful sequences from the Golay’s alphabet

- Several sequences of structuring elements \( \{B_{(i)}\} \) are useful in practice.
- Let us show two of them from the Golay alphabet (1969) for octagonal lattice. Structuring elements are displayed for the first two rotations, from which the other rotations can be derived.
- A concise representation of the composite structuring element in one matrix: the value 1 checks if the corresponding point (image pixel) is a subset of \( B_1 \) (objects). Simultaneously, the value 0 checks if the point is a subset of \( B_2 \) (background). Finally, the value * means that this element is not used in the matching process.
- Thinning and thickening sequential by Golay alphabet elements is idempotent.
Thinning by the element $L$, homotopic approximation of the width 1 skeleton

\[ L_1 = \begin{bmatrix} 0 & 0 & 0 \\ * & 1 & * \\ 1 & 1 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} * & 0 & 0 \\ 1 & 1 & 0 \\ * & 1 & * \end{bmatrix} \ldots \]

5 iterations
Thinning by the element L, cont.

Thinning until the idempotence is reached.
Cutting from free ends by the element $E$

$$E_1 = \begin{bmatrix} * & 1 & * \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ldots$$

If thinning by element $E$ is performed until the image does not change, then only closed contours remain.

5 iterations
Motivation for sequential morphology
Distance transformation

- It has not matter so far, in which order the morphological operation have been used in different location in the image. Operation could have been used in a random order, line after line, in parallel, . . .

- A more specific approach, in which the order of operator positions in the image is appropriately prescribed, can yield a substantial calculation speed up. The outcome of the operator will depend not only on the input image and the operator, but also on partial results of applying the operator previous locations in the image.

- By doing this, the useful global information can be accumulated, and the operator can explore previous results. The benefit is in gaining the speed and simplification of algorithms.

- Morphological operators based on the effective distance transform algorithm (dealt with earlier) constitute an important example of this approach, e.g. in calculating skeletons in binary morphology.
The distance transformation was explained in the lecture *Digital image, basic concepts*.
DT, starfish example, results

Consider a topographic view at the image function $f(x, y)$, i.e. as a landscape where the intensity corresponds to the altitude.

Skeleton points lie on the “mountain ridges”.

quazieuclidean Euclidean