# Geometry for robotics

Václav Hlaváč

Czech Technical University in Prague

Czech Institute of Informatics, Robotics and Cybernetics

160 00 Prague 6, Jugoslávských partyzánů 1580/3, Czech Republic

http://people.ciirc.cvut.cz/hlavac, vaclav.hlavac@cvut.cz

also Center for Machine Perception, http://cmp.felk.cvut.cz

*Courtesy: J. Xiao, R. Möller, T. Pajdla*

## Outline of the talk:

- Formalisms, notation, rehersal.
- Point, rotation in the 3D vector space.
- Rotation matrix, its inversion.
- Rotation and translation jointly.
- Euler, Cardan angles.
- Holonomic, non-holonomic robots.

# Where and why is geometry needed in robotics ?



- Motion in robotics is often approximated by a movement of a rigid body in a 3D space.
- We briefly review a needed mathematical formalism(s), i.e. geometry of motion.
- Three main application areas in robotics from a geometric point of view are:
  1. Open kinematic chain manipulators.
  2. Closed kinematic chain mechanisms.
  3. Mobile robots.

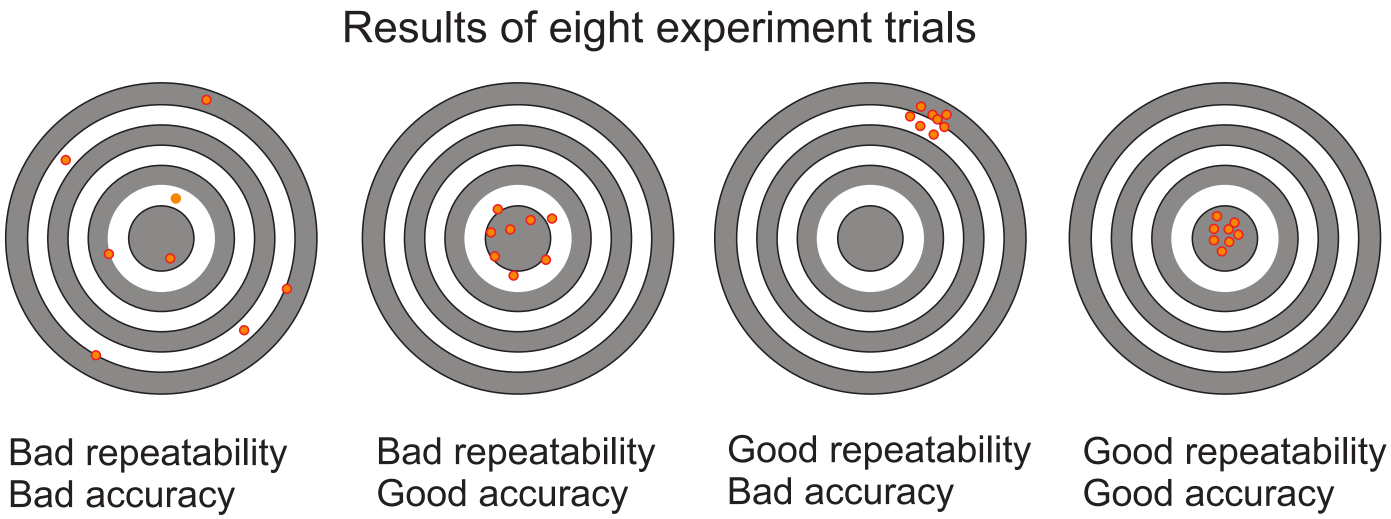
*The item 2 will not be tackled because it is too complicated for this overview course.*

# Accuracy and repeatability in robots



Let introduce these concepts informally.

- Accuracy is the difference (i.e. the error) between the requested result and the obtained result.
- Repeatability (e.g. of a robot) is a measure of its ability to achieve repetition of the same task.




Exact definitions are in ISO 9283: *Manipulating industrial robots – Performance criteria and related test methods.* 1998, last reviewed 2015.

# Formalisms

- Vector space.
- Projective space ($\Rightarrow$ homogeneous coordinates).
- *Quaternions. (not explained here)*

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We start with a quick math review.

# Notation



The notation of the subject B3M33PRO (Advanced robotics, lectured by Assoc. Prof. Tomas Pajdla for the Cybernetics and Robotics study branch in the coming semester) is used to maintain consistency.

| | | |
|---|---|---|
| $\vec{x}$ | ... | vector |
| $\mathsf{A}$ | ... | matrix |
| $\mathsf{A}_{ij}$ | ... | element $ij$ of $\mathsf{A}_{ij}$ |
| $\mathsf{A}^\top$ | ... | $\mathsf{A}$ transposed |
| $\lvert\mathsf{A}\rvert$ | ... | determinant of $\mathsf{A}$ |
| $\mathsf{I}$ | ... | identity matrix |
| $\mathsf{R}$ | ... | rotation matrix |
| $\vec{x} \times \vec{y}$ | ... | vector (cross) product of $\vec{x}, \vec{y}$ |

| | | |
|---|---|---|
| $\beta$ | ... | basis, the ordered triple $\beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3]$ of independent generator vectors |
| $\vec{x}_\beta$ | ... | column matrix of coordinates w.r.t. the basis $\beta$ |
| $\vec{x} \cdot \vec{y}$ | ... | scalar product of vectors $\vec{x}, \vec{y}$ |
| $\lVert\vec{x}\rVert$ | ... | Euclidean norm of $\vec{x}$, $\lVert\vec{x}\rVert = \sqrt{\vec{x} \cdot \vec{x}}$ |
| $\mathbb{R}$ | ... | real numbers |

# Dot product

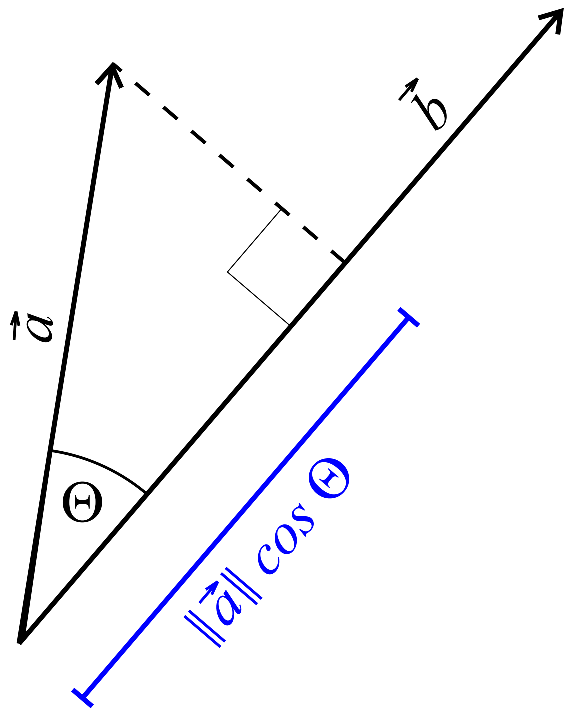
**Dot product** of vectors $\vec{a}, \vec{b}$
*(also scalar or inner product)*

- Geometric definition: $\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \, \cos\Theta$
- Algebraic definition: $\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i \, b_i$

*A two-dimensional example*

$\vec{a} = [a_x, a_y]^\top, \quad \vec{b} = [b_x, b_y]^\top$

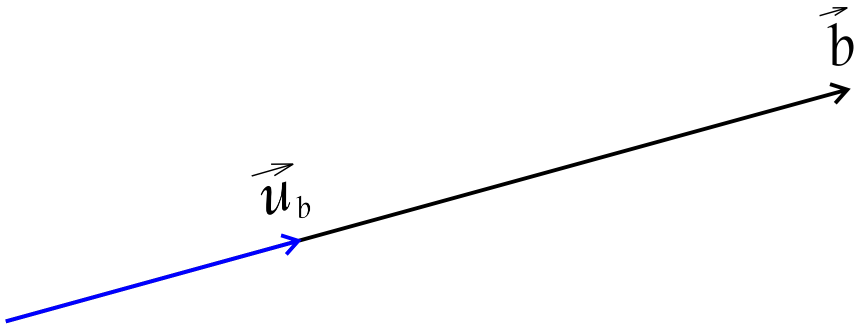
$$\vec{a} \cdot \vec{b} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} \cdot \begin{bmatrix} b_x \\ b_y \end{bmatrix} = a_x \, b_x + a_y \, b_y$$



# Unit vector

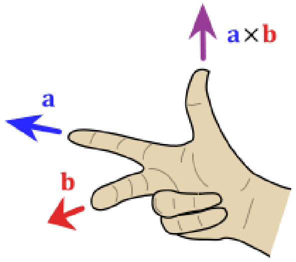
**Unit vector** $\vec{u}_b$ is a vector in the direction of a chosen vector (in our particular case of the vector $\vec{b}$), the magnitude of which equals to one.

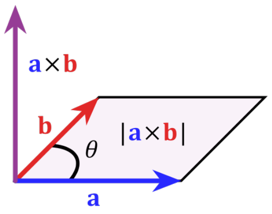
$$\vec{u}_b = \frac{\vec{b}}{||\vec{b}||}$$



# Cross (vector) product

- The cross product $\vec{a} \times \vec{b}$ is defined as a vector $\vec{c}$ that is perpendicular to both $\vec{a}$ and $\vec{b}$, with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span, i.e. $\|\vec{a}\|\,\|\vec{b}\|\sin\Theta$.
- Alternatively: $\vec{a} \times \vec{b} = \|\vec{a}\|\,\|\vec{b}\|\,\sin\Theta\,\vec{n}$ , where $\vec{n}$ is a unit vector perpendicular to the plane containing $\vec{a}$, $\vec{b}$ and the direction given by the right-hand rule.
- The cross-product is anti-commutative, i.e., $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$.

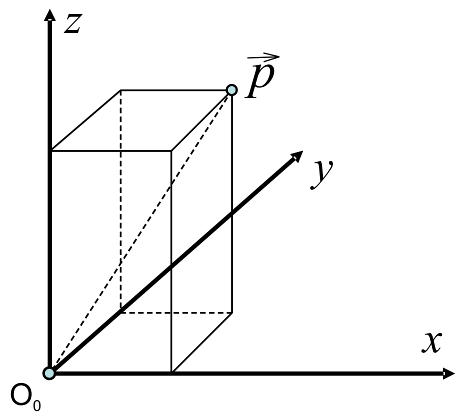



*Animation*

# Cartesian coordinate system

- Specifies the point in an $n$-dimensional Euclidean space. Coordinates are equal, up to the sign, to distances from the point to $n$ mutually perpendicular hyperplanes.
- In 3D, reference coordinate system $O_0xyz$.
- Point $\vec{p} = [p_x, p_y, p_z]^\top$ represented in $O_0xyz$:
  $\vec{p}_{xyz} = p_x\vec{\imath}_x + p_y\vec{\jmath}_y + p_z\vec{k}_z$
- $\vec{\imath} \cdot \vec{\jmath} = 0$, $\vec{\imath} \cdot \vec{k} = 0$, $\vec{k} \cdot \vec{\jmath} = 0$
  $|\vec{\imath}| = 1$, $|\vec{\jmath}| = 1$, $|\vec{k}| = 1$
- Name after René Descartes (latinized: Cartesius), who provided the first systematic link between Euclidean geometry and algebra.



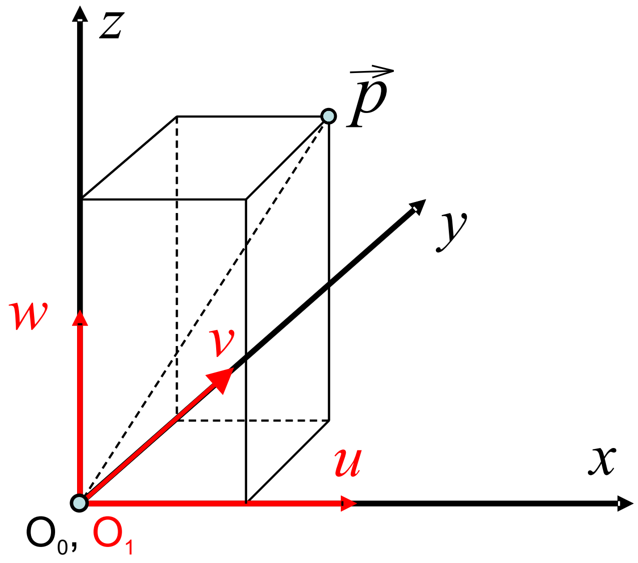

René Descartes, 1596-1650
Bílá hora battle (8. 11. 1620)
soldier on Catholic side.

# Reference coordinate system
# Representation of a point in it

- Reference coordinate system $O_0xyz$, unit coordinate vectors $\vec{x}$, $\vec{y}$, $\vec{z}$.

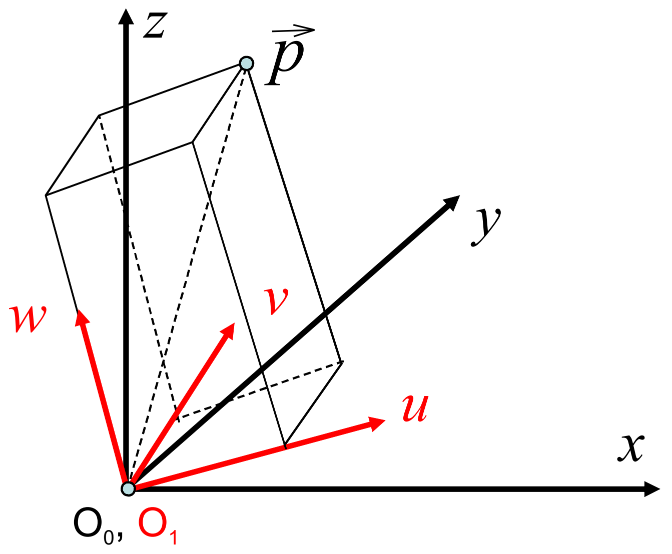
  Body attached frame $O_1uvw$, unit coordinate vectors $\vec{u}$, $\vec{v}$, $\vec{w}$.
- Point represented in $O_0xyz$: $\vec{p} = [p_x, p_y, p_z]^\top$

  $\vec{p}_{xyz} = p_x\vec{\imath}_x + p_y\vec{\jmath}_y + p_z\vec{k}_z$
- Point represented in $O_1uvw$: $\vec{p}_{uvw} = p_u\vec{\imath}_u + p_v\vec{\jmath}_v + p_w\vec{k}_w$
- If these two frames coincide then $p_u = p_x$, $p_v = p_y$, $p_w = p_z$



# Coordinate transformation, rotation only

- $\vec{p}_{xyz} = p_x \vec{\imath}_x + p_y \vec{\jmath}_y + p_z \vec{k}_z$
- $\vec{p}_{uvw} = p_u \vec{\imath}_u + p_v \vec{\jmath}_v + p_w \vec{k}_w$
- $\vec{p}_{xyz} = \mathsf{R}\, \vec{p}_{uvw}$, where R is a rotation matrix.
- $p_x$, $p_y$ and $p_z$ represent projections of a point $\vec{p}$ onto $O_0x$, $O_0y$, $O_0z$ axes, respectively.
- $p_x = \vec{\imath}_x \cdot \vec{p} = \vec{\imath}_x \cdot \vec{\imath}_u\, p_u + \vec{\imath}_x \cdot \vec{\jmath}_v\, p_v + \vec{\imath}_x \cdot \vec{k}_w\, p_w$
  $p_y = \vec{\imath}_y \cdot \vec{p} = \vec{\imath}_y \cdot \vec{\imath}_u\, p_u + \vec{\imath}_y \cdot \vec{\jmath}_v\, p_v + \vec{\imath}_y \cdot \vec{k}_w\, p_w$
  $p_z = \vec{\imath}_z \cdot \vec{p} = \vec{\imath}_z \cdot \vec{\imath}_u\, p_u + \vec{\imath}_z \cdot \vec{\jmath}_v\, p_v + \vec{\imath}_z \cdot \vec{k}_w\, p_w$



# Rotation matrix

- Repeated from the previous slide:

$$p_x = \vec{\imath}_x \cdot \vec{p} = \vec{\imath}_x \cdot \vec{\imath}_u \, p_u + \vec{\imath}_x \cdot \vec{\jmath}_v \, p_v + \vec{\imath}_x \cdot \vec{k}_w \, p_w$$
$$p_y = \vec{\imath}_y \cdot \vec{p} = \vec{\imath}_y \cdot \vec{\imath}_u \, p_u + \vec{\imath}_y \cdot \vec{\jmath}_v \, p_v + \vec{\imath}_y \cdot \vec{k}_w \, p_w$$
$$p_z = \vec{\imath}_z \cdot \vec{p} = \vec{\imath}_z \cdot \vec{\imath}_u \, p_u + \vec{\imath}_z \cdot \vec{\jmath}_v \, p_v + \vec{\imath}_z \cdot \vec{k}_w \, p_w$$

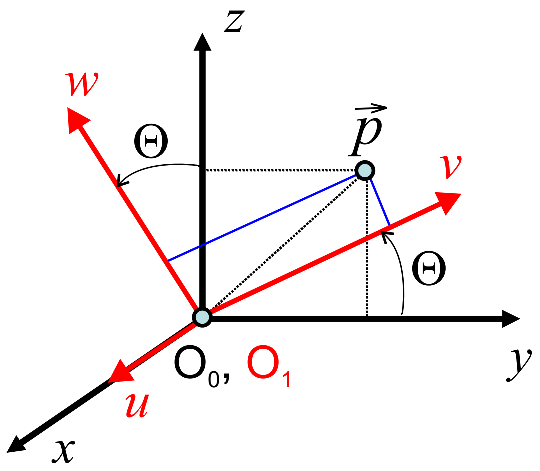
- Expressed as a matrix multiplication:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \vec{\imath}_x \cdot \vec{\imath}_u & \vec{\imath}_x \cdot \vec{\jmath}_v & \vec{\imath}_x \cdot \vec{k}_w \\ \vec{\jmath}_y \cdot \vec{\imath}_u & \vec{\jmath}_y \cdot \vec{\jmath}_v & \vec{\jmath}_y \cdot \vec{k}_w \\ \vec{k}_z \cdot \vec{\imath}_u & \vec{k}_z \cdot \vec{\jmath}_v & \vec{k}_z \cdot \vec{k}_w \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$

- Example, rotation about axis $x$ by $\Theta$:

$$\mathsf{R} = \mathsf{R}(x, \Theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta \\ 0 & \sin\Theta & \cos\Theta \end{bmatrix}$$

Example, rotation about axis $x$ by the angle $\Theta$:




$$p_x = p_u$$
$$p_y = p_v \cos\Theta - p_w \sin\Theta$$
$$p_z = p_v \sin\theta + p_w \cos\Theta$$

# Rotation about coordinate axes

- Rotation about axis $x$ by $\Theta$:

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \; \vec{v} = \begin{bmatrix} 0 \\ \cos\Theta \\ \sin\Theta \end{bmatrix} ; \; \vec{w} = \begin{bmatrix} 0 \\ -\sin\Theta \\ \cos\Theta \end{bmatrix} ; \quad \mathsf{R} = \mathsf{R}(x,\Theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta \\ 0 & \sin\Theta & \cos\Theta \end{bmatrix}$$

- Rotation about axis $y$ by $\Theta$:

$$\mathsf{R} = \mathsf{R}(y,\Theta) = \begin{bmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & \cos\Theta \end{bmatrix}$$

- Rotation about axis $z$ by $\Theta$:

$$\mathsf{R} = \mathsf{R}(z,\Theta) = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Inverting rotation matrix

- $\vec{p}_{xyz} = \mathsf{R}\,\vec{p}_{uvw}$

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \vec{\imath}_x \cdot \vec{\imath}_u & \vec{\imath}_x \cdot \vec{\jmath}_v & \vec{\imath}_x \cdot \vec{k}_w \\ \vec{\jmath}_y \cdot \vec{\imath}_u & \vec{\jmath}_y \cdot \vec{\jmath}_v & \vec{\jmath}_y \cdot \vec{k}_w \\ \vec{k}_z \cdot \vec{\imath}_u & \vec{k}_z \cdot \vec{\jmath}_v & \vec{k}_z \cdot \vec{k}_w \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$

- $\vec{p}_{uvw} = \mathsf{Q}\,\vec{p}_{xyz}$. Notice: The dot product is commutative.

$$\begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix} = \begin{bmatrix} \vec{\imath}_u \cdot \vec{\imath}_x & \vec{\imath}_u \cdot \vec{\jmath}_y & \vec{\imath}_u \cdot \vec{k}_z \\ \vec{\jmath}_v \cdot \vec{\imath}_x & \vec{\jmath}_v \cdot \vec{\jmath}_y & \vec{\jmath}_v \cdot \vec{k}_z \\ \vec{k}_w \cdot \vec{\imath}_x & \vec{k}_w \cdot \vec{\jmath}_y & \vec{k}_w \cdot \vec{k}_z \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

- Rotation matrices are orthogonal, i.e.
  $\mathsf{Q} = \mathsf{R}^{-1} = \mathsf{R}^\top \Rightarrow \mathsf{QR} = \mathsf{R}^\top\mathsf{R} = \mathsf{RR}^\top = \mathsf{R}^{-1}\mathsf{R} = \mathsf{I}$.
  (a) Column vector are mutually perpendicular unit vectors; (b) $\det \mathsf{R} = \pm 1$ ($+1$ for right-hand coordinates); (c) $\mathsf{R} \in \mathrm{SO}(3)$, i.e. special orthogonal group of rotational matrices of the third order *(to be explained soon)*.

# Composite rotation matrix

- A sequence of finite rotations.
- Matrix multiplications do not commute $\Rightarrow$ the correct order is important.
- Point $\vec{p}$ is represented as $\vec{p}_0$ w.r.t. to its coordinates $Oi_0j_0k_0$.
  Point $\vec{p}_1$ similarly as $\vec{p}_1$ w.r.t. $Oi_1j_1k_1$.
  Point $\vec{p}_2$ similarly as $\vec{p}_2$ w.r.t. $Oi_2j_2k_2$.
- $\vec{p}_0 = \mathsf{R}_0^1\,\vec{p}_1$ and $\vec{p}_1 = \mathsf{R}_1^2\,\vec{p}_2$
- $\mathsf{R}_0^2 = \mathsf{R}_0^1\,\mathsf{R}_1^2$, consequently $\vec{p}_0 = \mathsf{R}_0^2\,\vec{p}_2$

$$\begin{tikzcd} p_2 \arrow[r, "R_1^2"] \arrow[rr, bend right, "R_0^1\,R_1^2"'] & p_1 \arrow[r, "R_0^1"] & p_0 \end{tikzcd}$$

# Example, a composite rotation, around $z$-axis first, around $y$ axis next

1. Rotation around the current $z$-axis by the angle $\Theta$.

2. Rotation around the current $y$-axis by the angle $\Phi$.

$$\mathsf{R} = \mathsf{R}(y,\Phi)\,\mathsf{R}(z,\Theta) = \begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ -\sin\Phi & 0 & \cos\Phi \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Phi\ \cos\Theta & -\cos\Phi\ \sin\Theta & \sin\Phi \\ \sin\Theta & \cos\Theta & 0 \\ -\sin\Phi\ \cos\Theta & \sin\Phi\ \sin\Theta & \cos\Phi \end{bmatrix}$$

# Example, a composite rotation, around $y$-axis first, around $z$-axis next



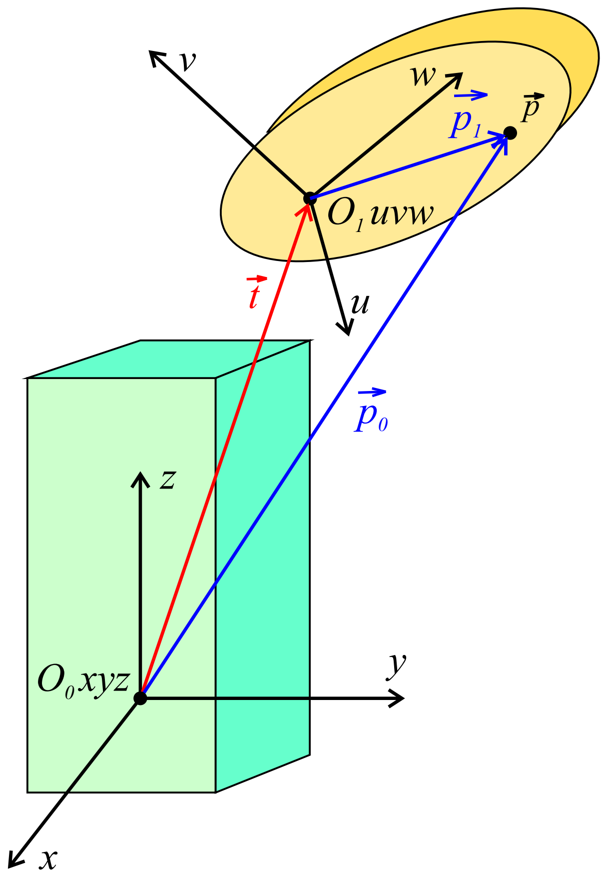
1. Rotation around the current $y$-axis by the angle $\Phi$.
2. Rotation around the current $z$-axis by the angle $\Theta$.

$$\mathsf{R} = \mathsf{R}(z,\Theta)\,\mathsf{R}(y,\Phi) = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ -\sin\Phi & 0 & \cos\Phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Theta\,\cos\Phi & -\sin\Theta & \cos\Theta\,\sin\Phi \\ \sin\Theta\,\cos\Phi & \cos\Theta & 0 \\ -\sin\Phi & 0 & \cos\Theta \end{bmatrix}$$

# Rotation and translation jointly

- A point (vector) $\vec{p}$ originally expressed with respect to the coordinate system $O_1uvw$ as $\vec{p}_1$ is newly represented with respect to the coordinate system $O_0xyz$ as $\vec{p}_0$.
- The transformation writes as $\vec{p}_0 = \mathrm{R}\,\vec{p}_1 + \vec{t}$, where R is the rotation matrix aligning the coordinate system $O_0xyz$ to $O_1uvw$ and $\vec{t}$ is a translation vector bringing the origin $O_0$ to the origin $O_1$.
- It is of advantage to express the rotation and translation as a matrix operation. This requires introducing homogeneous coordinates, i.e. embedding into a projective space.



# Projective space

- It is of advantage to embed the joint rotation and translation into a projective space $\mathbb{P}^d$.
- Consider $(d+1)$-dimensional vector space without its origin, $\mathbb{R}^{d+1} - \{(0,\ldots,0)\}$.
- Consider the equivalence relation

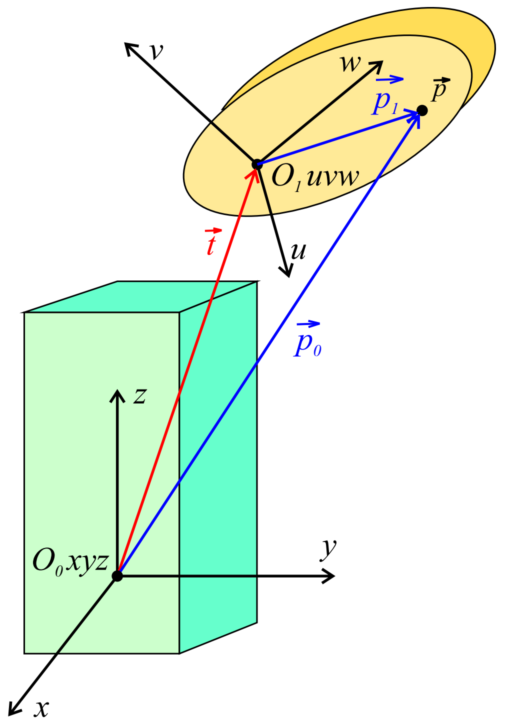
$$\begin{array}{rcccc} & [x_1,\ldots,x_{d+1}]^\top & \equiv & & [x'_1,\ldots,x'_{d+1}]^\top \\ \text{iff } \exists\, \alpha \neq 0 : & [x_1,\ldots,x_{d+1}]^\top & = & \alpha & [x'_1,\ldots,x'_{d+1}]^\top \end{array}$$

- The projective space $\mathbb{P}^d$ is the quotient space of this equivalence relation.
- Points in the projective space are expressed in homogeneous co-ordinates (called also projective coordinates) $\vec{x} = [x'_1,\ldots,x'_d,1]^\top$.

# Homogeneous transformation

- In non-homogeneous coordinates, cf. slide 18:
  A point (vector) $\vec{p}$ originally expressed with respect to the coordinate system $O_1uvw$ as $\vec{p}_1$ is newly represented with respect to the coordinate system $O_0xyz$ as $\vec{p}_0$ as
  $\vec{p}_0 = \mathsf{R}\,\vec{p}_1 + \vec{t}.$
- Express $\vec{p}_0$, $\vec{p}_1$ in homogeneous coordinates as $\vec{p}_{0h}$, $\vec{p}_{1h}$.
- The joint rotation and translation can be written in the matrix form

$$\vec{p}_{0h} = \left[\begin{array}{ccc|c} & \mathsf{R} & & \vec{t} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \vec{p}_{1h}$$




When it is obvious that we deal with homogeneous coordiantes, we omit subscripts $_h$.

# Group (algebraic structure)

- It is useful to express properties of rotations in a more abstract way. E.g., we will use it in this course later when dealing with the (robot) configuration space.
- The group (the algebraic structure)
  A set $G$ together with a binary operation $\circ$ on elements of $G$ is a group if it satisfies axioms

  1. *Closure:* If $g_1, g_2 \in G$, then $g_1 \circ g_2 \in G$.
  2. *Identity:* The identity element $e$ exists such that $g \circ e = e \circ g = g$ for every $g \in G$.
  3. *Inverse:* For each $g \in G$ there exists a (unique) inverse $g^{-1} \in G$, such that $g \circ g^{-1} = g^{-1} \circ g = e$.
  4. *Associativity:* If $g_1, g_2, g_3 \in G$, then $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

*Note: Closure: A set is closed under an operation if performance of that operation on members of the set always produces a member of that set.*

# Euclidean group

- In mathematics, the elements of Euclidean group $\mathrm{E}(n)$ are the isometries associated with the Euclidean distance, and are called Euclidean isometries, Euclidean transformations or rigid transformations.
- Euclidean transformations decompose into direct isometries and indirect isometries, an indirect isometry being an isometry that transforms any object into its mirror image.
- The direct Euclidean isometries form a group, the special Euclidean group $\mathrm{SE}(n)$, whose elements are called Euclidean motions or rigid motions.
- The Euclidean group for $\mathrm{SE}(3)$ is used for the kinematics of a rigid body, in classical mechanics. Consider a rigid body described in reference frame $O_0xyz$:

$$\mathrm{SE}(3) = \left\{ \left[ \begin{array}{ccc|c} & \mathsf{R} & & \vec{t} \\ \hline 0 & 0 & 0 & 1 \end{array} \right],\ \mathsf{R} \in \mathsf{R}^{3\times 3},\ \vec{t} \in \mathbb{R}^3,\ \mathsf{R}^\top \mathsf{R} = \mathsf{R}\mathsf{R}^\top = \mathsf{I},\ |\mathsf{R}| = 1 \right\}$$

*Note: An isometry (or congruence, or congruent transformation) is a distance-preserving transformation between metric spaces, usually assumed to be bijective.*

# Special orthogonal group (1)

- Orthogonal group in dimension $n$, denoted $\mathrm{O}(n)$, is the group of $n \times n$ orthogonal matrices, where the group operation is the matrix multiplication. An orthogonal matrix is a real matrix whose inverse equals to its transpose.
- An important subgroup of $\mathrm{O}(n)$ is the special orthogonal group $\mathrm{SO}(n)$ of $n \times n$ orthogonal matrices of determinant 1.

  Because the determinant of an orthogonal matrix is either $1$ or $-1$, and so the orthogonal group has two components. The component containing the identity 1 is the special orthogonal group $\mathrm{SO}(n)$.
- $\mathrm{SO}(n)$ is also called the rotational group because its elements are usual rotations around a point (in dimension 2) or a line (in dimension 3), cf. $\mathrm{SO}(2)$ and $\mathrm{SO}(3)$.

# Special orthogonal group (2)

- $\mathrm{SO}(2)$, the circle group. One way to think about the circle group is that it describes how to add angles, where only angles between $0^\circ$ and $360^\circ$ are permitted.
- $\mathrm{SO}(3)$, the 3D rotation group, is the group of all rotations about the origin of 3D Euclidean space $\mathbb{R}^3$ under the composition operation.

# Euler angles, the minimal representation

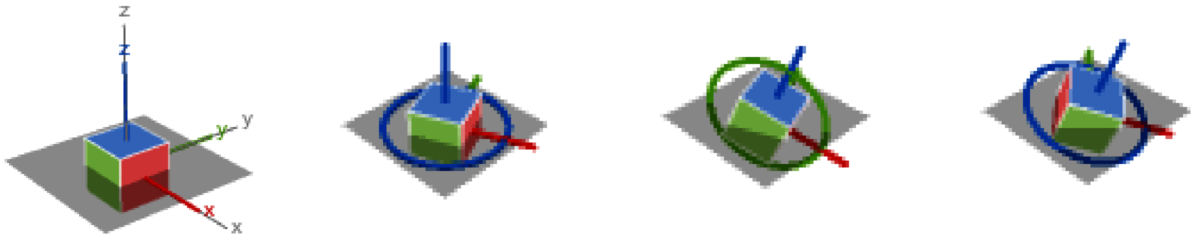
- Rotation matrices provide a redundant representation of the frame orientation. They are given by nine elements.
- These elements are not independent because they are related (a) by the orthogonality condition $\mathrm{R}^\top \mathrm{R} = \mathrm{I}$ $(3$ constraints), unitary relationship $(3$ constraints), and $\det(R) = 1$.
- This implies that three parameters $(9 - 3 - 3 = 3)$ suffice to express orientation of a rigid body in 3D space.
- Orientation expressed by three parameters constitutes a minimal representation.
- There are 12 possible sequences of rotation axes, divided into two groups:
  - Euler angles: $z\,x\,z$, $x\,y\,x$, $y\,z\,y$, $z\,y\,z$, $x\,z\,x$, $y\,x\,y$
  - Cardan angles (after Jerome Cardan or Gerolamo Cardano, also called Tait-Brian, nautical, yaw-pitch-roll):

    $x\,y\,z$, $y\,z\,x$, $z\,x\,y$, $x\,z\,y$, $z\,y\,x$, $y\,x\,z$

# $z\,y\,z$ Euler angles

Composition of three elementary rotations

- Rotate the reference frame by the angle $\phi$ about $z$-axis.
- Rotate the current frame by the $\theta$ about (tranformed) axis $y'$.
- Rotate the current frame by the angle $\psi$ about axis $z''$.




Input $\qquad \phi \qquad \theta \qquad \psi$

# $z\,y\,z$ Euler angles, rotation matrices

The rotation described by $z\;y\;z$ composes three rotations of the current frame $R = R_z(\phi)\,R_{y'}(\theta)\,R_{z''}(\psi)$.

- Rotation by the angle $\phi$ around axis $z$: $R_z = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Rotation by the angle $\theta$ around axis $y'$: $R_{y'} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
- Rotation by the angle $\psi$ around axis $z''$: $R_{z''} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

# $z\,y\,z$ Euler angles, the direct solution

Given: Three $z\;y\;z$ Euler angles.

Task: Rotate (1) by the angle $\phi$ along the axis $z$ giving the new axes $x'$, $y'$ and $z' \equiv z$; (2) by the angle $\theta$ along the axis $y'$ giving new axes $x''$, $y''$, $z''$ and (3) by the angle $\psi$ around the axis $z''$.

Outcome: The rotation matrix R.

$$\mathsf{R} = \mathsf{R}_z(\phi)\,\mathsf{R}_{y'}(\theta)\,\mathsf{R}_{z''}(\psi) =$$

$$= \begin{bmatrix} c_\phi\, c_\theta\, c_\psi - s_\phi\, s_\psi\,, & -c_\phi\, c_\theta\, s_\psi - s_\phi\, c_\psi\,, & c_\phi\, s_\theta \\ s_\phi\, c_\theta\, c_\psi + c_\phi\, s_\psi\,, & -s_\phi\, c_\theta\, s_\psi + c_\phi\, c_\psi\,, & s_\phi\, s_\theta \\ -s_\theta\, c_\psi\,, & s_\theta\, s_\psi\,, & c_\theta \end{bmatrix}$$

# $z\,y\,z$ Euler angles, the inverse solution

$$\mathsf{R} = \begin{bmatrix} r_{11}, & r_{12}, & r_{13} \\ r_{21}, & r_{22}, & r_{23} \\ r_{31}, & r_{32}, & r_{33} \end{bmatrix} = \begin{bmatrix} c_\phi\, c_\theta\, c_\psi - s_\phi\, s_\psi\,, & -c_\phi\, c_\theta\, s_\psi - s_\phi\, c_\psi\,, & c_\phi\, s_\theta \\ s_\phi\, c_\theta\, c_\psi + c_\phi\, s_\psi\,, & -s_\phi\, c_\theta\, s_\psi + c_\phi\, c_\psi\,, & s_\phi\, s_\theta \\ -s_\theta\, c_\psi\,, & s_\theta\, s_\psi\,, & c_\theta \end{bmatrix}$$

The solution to the inverse problem, i.e. calculating Euler angles from the rotation matrix R, is given by explicit formulas as

- $\theta = \cos^{-1}(r_{33})$ because $r_{33} = \cos\theta$
- $\phi = \tan^{-1}\left(\frac{r_{23}}{r_{13}}\right)$ because $r_{13} = \cos\phi\,\sin\theta$; $r_{23} = \sin\phi\,\sin\theta$
  $\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{r_{23}}{\sin\theta} / \frac{r_{13}}{\sin\theta} = \frac{r_{23}}{r_{13}}$
- $\psi = \tan^{-1}\left(\frac{r_{32}}{-r_{31}}\right)$ because $r_{31} = -\sin\theta\,\sin\psi$; $r_{32} = \sin\theta\,\sin\psi$
  analogically to $\phi$

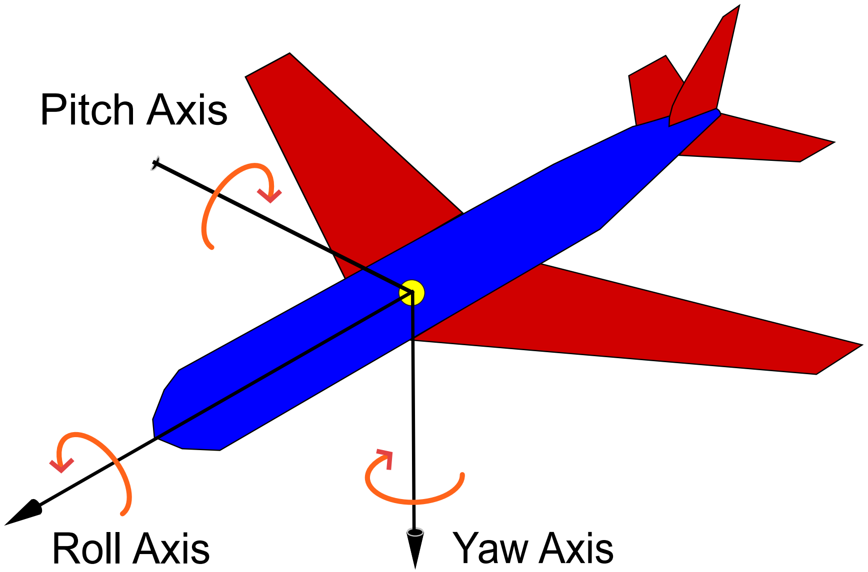
*Note: A little more care is needed due to multiple solutions and singularities in practice.*

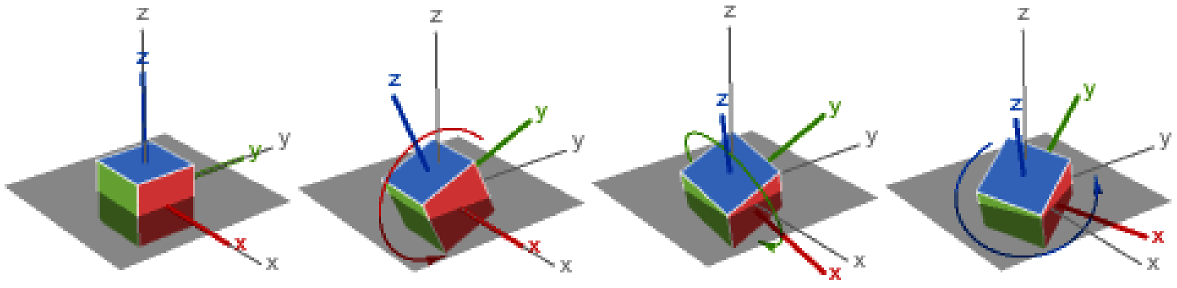
# $x\ y\ z$ Cardan angles, yaw-pitch-roll



Composition of three elementary rotations

- Rotate the reference frame by the angle $\psi$ about $x$-axis (yaw, Czech zatáčení).
- Rotate the reference frame by the angle $\theta$ about axis $y'$ (pitch, Czech podélný sklon).
- Rotate the reference frame by the angle $\phi$ about axis $z''$ (roll, Czech příčný náklon).



# Holonomicity in robotics

- Holonomicity refers to the relationship between the controllable and total degrees of freedom of a given robot (or part thereof).
- Holonomic: if the controllable degrees of freedom is equal or greater to the total degrees of freedom.
- Non-holonomic: if the controllable degrees of freedom are less than the total degrees of freedom.
- Redundant robot: if it has more controllable degrees of freedom than degrees of freedom in its task space.

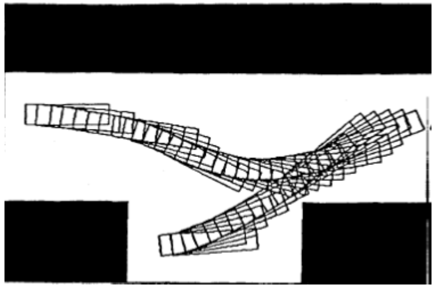
# Example: A car = non-holonomic

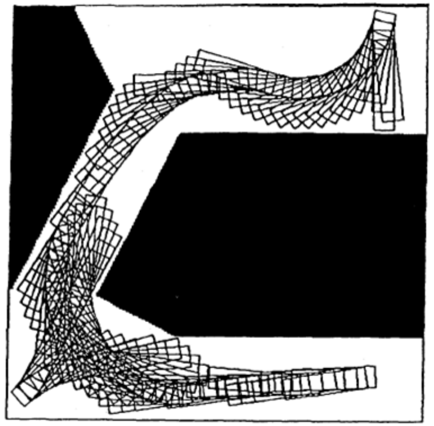
- Three degrees of freedom: its position in two axes, and its orientation relative to a fixed heading.
- Only two controllable degrees of freedom: acceleration/braking and the angle of the steering wheel.
- A car heading (the direction, in which it is traveling) must remain aligned with the orientation of the car, or $180^o$ from it if the car is in reverse. The car has no other allowable direction, assuming there is no skidding or sliding.
- Thus, not every path in the space is achievable.

# Approximation by a holonomic trajectory



- For a car, not every trajectory in the space is achievable.
- However, every trajectory can be approximated by a holonomic trajectory – this is called a (dense) homotopy principle (comes from mathematics, differential equations), cf. https://en.wikipedia.org/wiki/Homotopy_principle.
- The non-holonomicity of a car makes parallel parking and turning in the road difficult, but the homotopy principle says that these are always possible, assuming that clearance exists.





# A human arm is holonomic

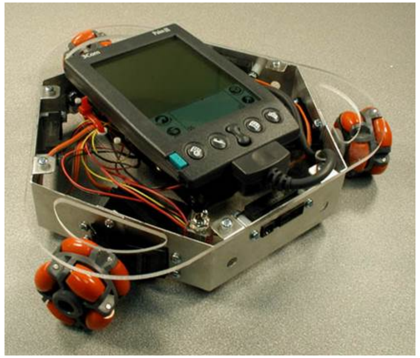
- A human arm is holonomic and redundant.
- It is a redundant system because it has 7 degrees of freedom (3 in the shoulder - rotations about each axis, 2 in the elbow - bending and rotation about the lower arm axis, and 2 in the wrist, bending up and down (i.e. pitch), and left and right (i.e. yaw)).
- There are only 6 physical degrees of freedom in the task of placing the hand (x, y, z, roll, pitch and yaw), while fixing the seven degrees of freedom fixes the hand.

# Holonomic locomotion

- Holonomic forms of locomotion allow vehicles to immediately move in any direction without needing to turn first.
- *Example:*

  Mecanum (Sweedish) wheel, e.g Holbot (by a FEL ČVUT student Igor Kruhák, 2003).
- *Counterexample:*

  Segway (inverted pendulum principle) is not holonomic. It has 2 DOFs. There are 3 DOFs to place, orient it in the environment.



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