Robot trajectory generation

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Courtesy: Alessandro De Luca, Claudio Melchiorri; other presentations from the web.

Outline of the talk:

- Trajectory × path.
- Trajectory generation, problem formulation.
- Trajectory in operational and joint spaces.
- Curve approximation.
- Trajectory classification; Dubins curves.
- Trajectory approximation by polynomials.

Practical requirements related to trajectories

Note: We consider a robotic manipulator (open kinematic chain) for simplicity. Generalization of methods for mobile robots is not difficult.

- Endow the robot with the capability to move the manipulator arm and its end effector or the mobile robot from the initial posture to the final posture.
- Motion laws have to be considered in order not to:
  - violate saturation limits of joint drives;
  - excite the resonant modes of the actuator mechanical structure, being driven by electric/hydraulic/pneumatic or other more exotic drives.
- Explore path planning and trajectory generation methods providing smooth trajectories solving the practical robotics task.

Question: Why are smooth trajectories preferred over jerky ones?

Terminology: path vs. trajectory

- *Note: Terms path, trajectory are often exchanged because they are perceived as synonyms informally.*
- **Path** consists of ordered locii of points in the space (either the joint space or the operational space), which the robot should follow.
  - The path provides a pure geometric description of motion.
  - The path is usually planned globally taking into account obstacle avoidance, traversing a complicated maze, etc.
- **Trajectory** is a path plus velocities and accelerations in its each point.
  - The design of a trajectory does not need global information, which simplifies the task significantly.
  - The trajectory is specified and designed locally (divide and conquer methodology). Parts of the path are covered by individual trajectories.
  - It is often required that pieces of trajectories join smoothly, which induces that a single trajectory design takes into account only neighboring trajectories from the path.

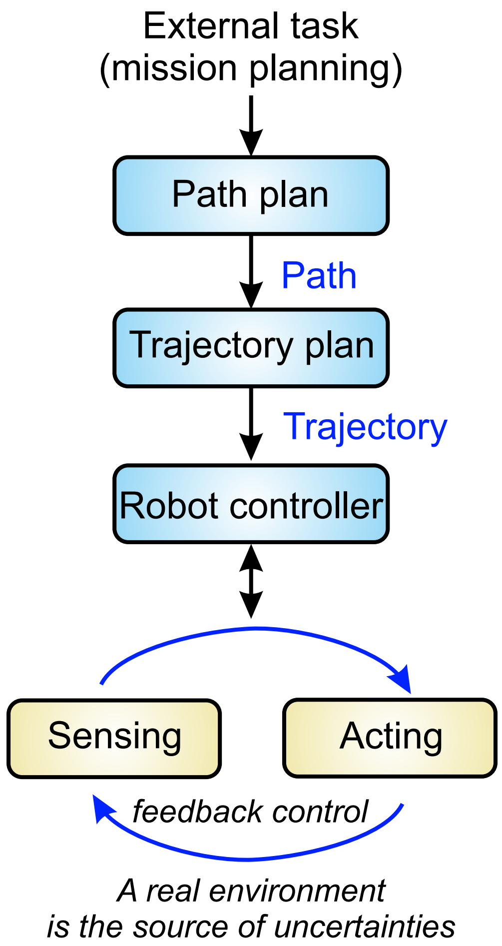
Robot motion planning, an overview

Path planning (global)

- The (geometric) path is a sequence of waypoints defining the trajectory coarsely.
- Issues solved at this level: obstacle avoidance, shortest path.

Trajectory generating (local)

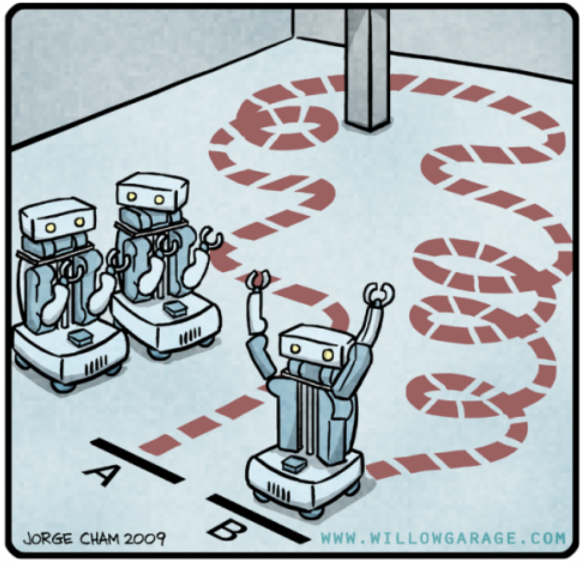
- The path provided by path planning constitutes the input to the trajectory generator.
- Trajectory generator "approximates" the desired path waypoints by a class of polynomial functions and
- generates a time-based control sequence moving the manipulator/mobile platform from its initial configuration to the destination.



Path planning, the problem formulation

- The path planning task:
  Find a collision free path for the robot from one configuration to another configuration.
- Path planning is an algorithmically difficult search problem.
  - The involved task has an exponential complexity with respect to the degrees of freedom (controllable joints).
  - With industrial robots, the path planning is often replaced by a path/trajectory taught in by a human operator.

R.O.B.O.T. Comics



"HIS PATH-PLANNING MAY BE SUB-OPTIMAL, BUT IT'S GOT FLAIR."

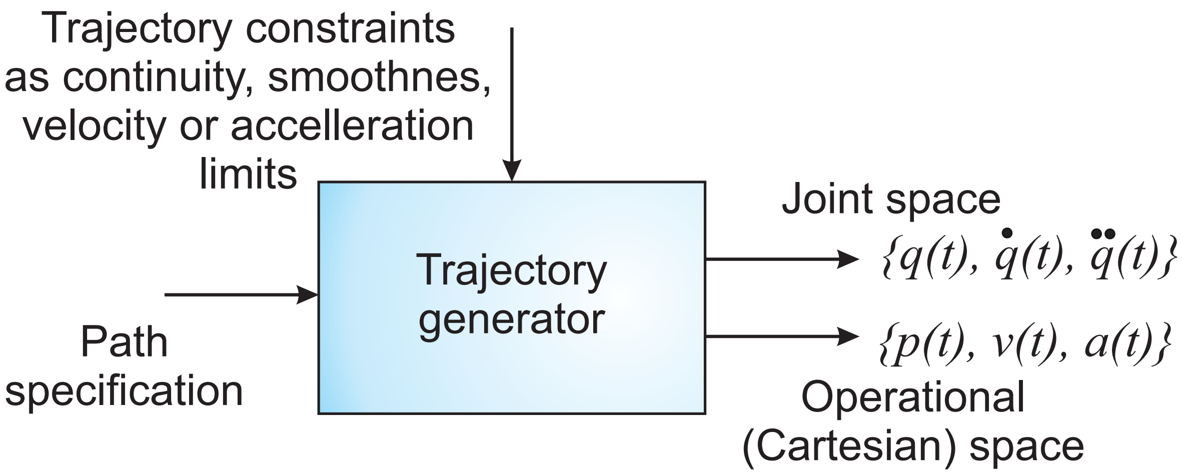
Note: A separate (next) lecture will be devoted to the path planning.

Trajectory generation, the problem formulation

- Trajectory generation aims at creating inputs to the motion control system, which ensures that the planned trajectory is smoothly executed.
- The planned path is typically represented by way-points, which is the sequence of points (or end-effector poses) along the path.
- Trajectory generating = creating a trajectory connecting two or more way-points.
  - In the industrial settings, a trajectory is taught-in by a human expert and later played back (by teach-and-playback).

    A more recent approach utilizes several tens of trajectories performed by human experts as the input. They vary statistically. Machine learning techniques are used to create the final trajectory.
  - In general, e.g., with mobile robots and more and more with industrial robots, the path points generated by the path planner are smoothly approximated using function approximation methods from mathematics.

Trajectory generating, illustration



- Trajectory generating $=$ finding the desired joint space trajectory $\mathbf{q}(t)$ given the desired operational (Cartesian) path inverse kinematics.
- $\mathbf{q}$ is a vector of joint parameters. Its dimension matches to the number of DOFs.
- $\mathbf{p}(t) = (x(t), y(t), z(t))$ is the position, $\mathbf{v}(t) = (x'(t), y'(t), z'(t)$ is the velocity, $\mathbf{a}(t) = (x''(t), y''(t), z''(t)$ is the acceleration.

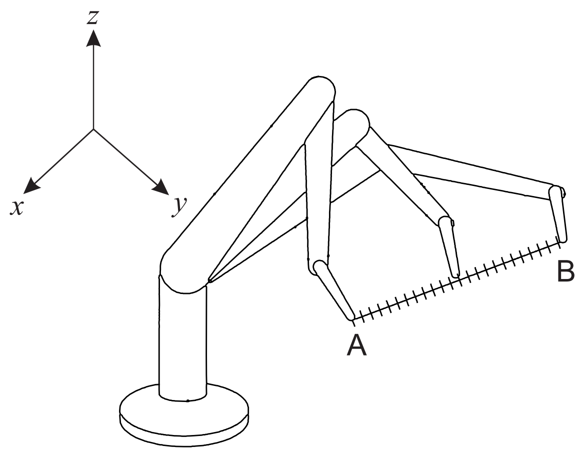
Joint space vs. operational space

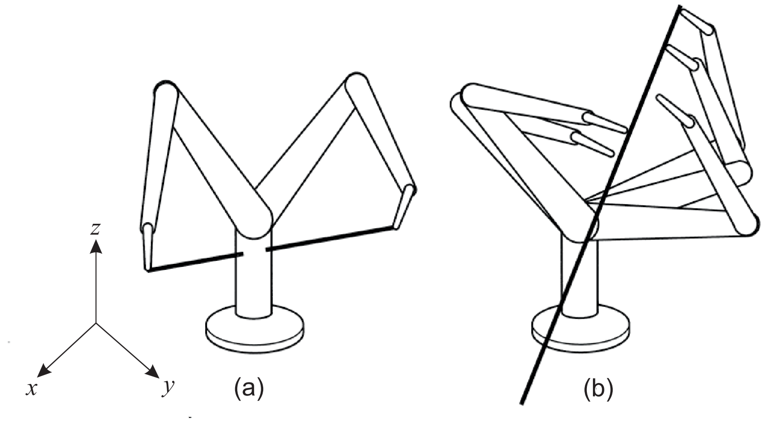
- Joint-space description:
  - The description of the motion to be made by the robot by its joint values.
  - The motion between the two points is unpredictable.
- Operational space description:
  - Operational space = (Cartesian space or working space) in many cases.
  - The motion between the two points is known and controllable at all times.
  - It is easy to visualize the trajectory, but it is difficult to ensure that singularity does not occur.

The path planning/trajectory generation can be done in both the joint space or operational spaces.

Example: Troubles with the operational (Cartesian) space

Task: Create a trajectory of the manipulator end effector to follow a straight line.





(a) The trajectory specified in Cartesian coordinates may force the robot to run into itself, see Figure (a) above;

(b) The trajectory may requires a sudden change in the joint angles due to singularities, see Figure (b) above.

Trajectory in the operational space

- Calculate the path from the initial point to the final point.
- Assign a total time T_{path} to traverse the path.
- Discretize points in time and space.
- Blend a continuous time function between these points.
- Solve inverse kinematics at each step.

Advantages

- Collision free path can be obtained.

Disadvantages

- Computationally expensive due to involved inverse kinematics.
- It is unknown how to set the total time T_{path}.

Trajectory in the joint space

- Calculate the inverse kinematics solution from the initial point to the final point.
- Assign the total time T_{path} using maximal velocities in joints.
- Discretize the individual joint trajectories in time.
- Blend a continuous function between these point.

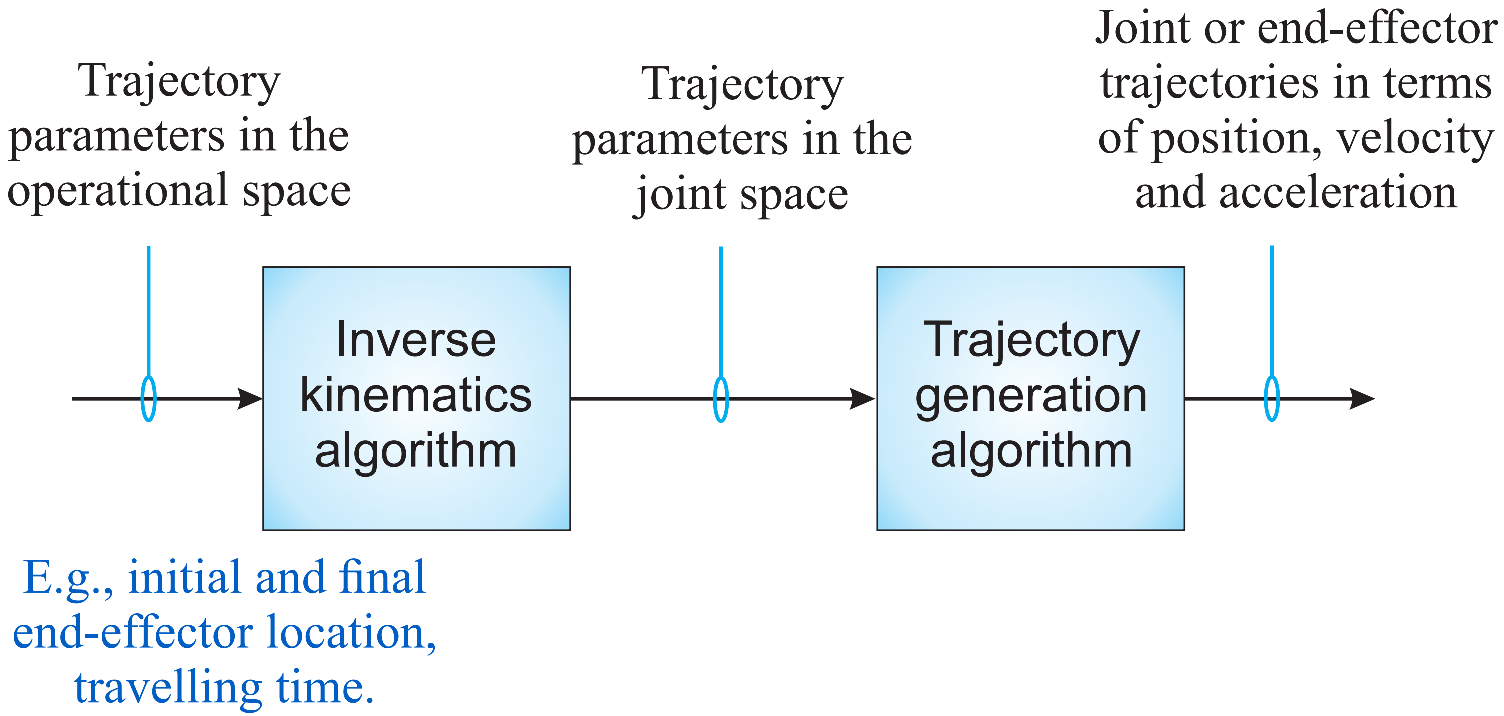
Advantages

- The inverse kinematics is computed only once.
- The approach can take into account joint angles limits and velocity constraints easily.

Disadvantages

- It is more difficult to deal with obstacles represented in the operational space. Nevertheless, it is doable.

Trajectory from the operational space to the joint space



Types of the trajectory control

- Displacement control = control the end effector/mobile robot displacement, i.e., angles or positions in space, maybe including dynamics of motion.

  Examples:

  - Moving payloads.
  - Painting objects.

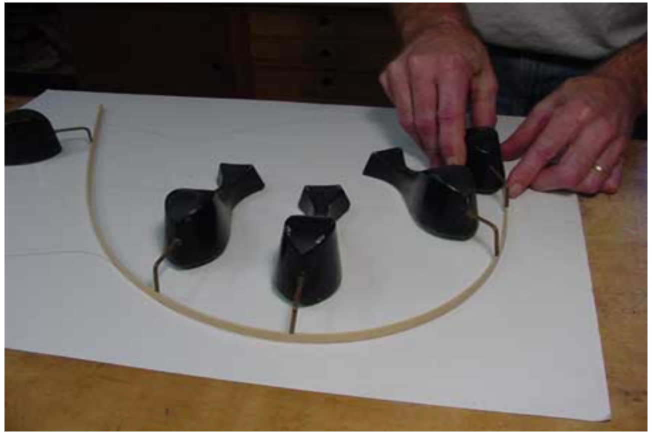
- Force control = control of both displacement (as above) and applied force.

  Examples:

  - Machining.
  - Grinding.
  - Sanding.

Curve approximation, the motivation

- A draftsman uses "ducks" and strips of wood (splines) to draw curves.
- "Wooden splines" provide the second-order continuity
- and pass through control points.



Examples of functions used for trajectory interpolation:

- Polynomials of different orders.
- Linear functions with parabolic blends.
- Splines.

Illustration of trajectory generation issues

Different approaches will be demonstrated on a simple example.

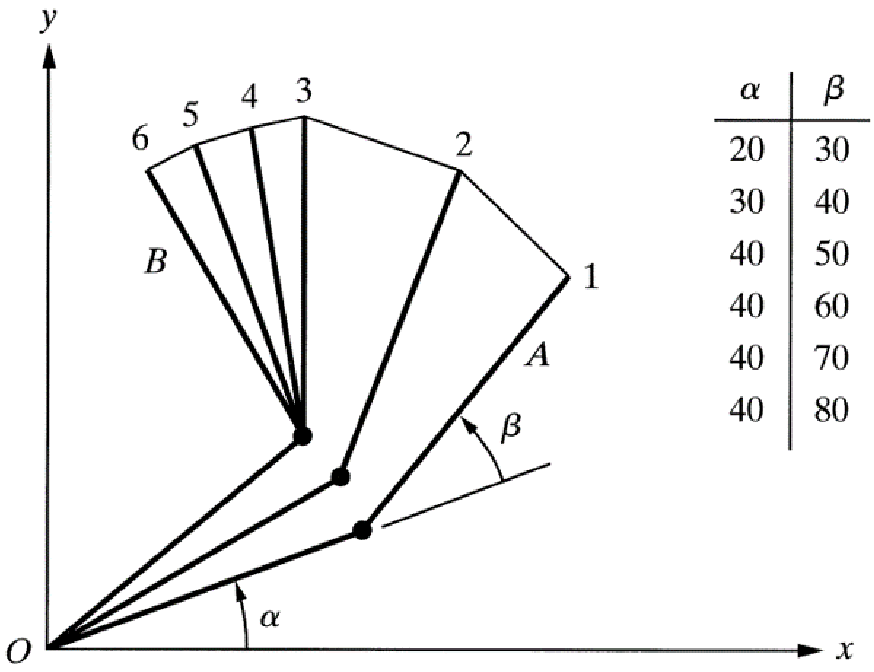
- Let us consider a simple two degrees of freedom robot.
- We desire to move the robot from Point A to Point B.
- Let us assume that both joints of the robot can move at the maximum rate of 10 degree/sec.

Note:

- Jerk (informally) = thrashing of the mechanism.
- Jerk (mechanics, formally) = the rate of velocity (time derivative of a trajectory) change.

Non-normalized trajectory

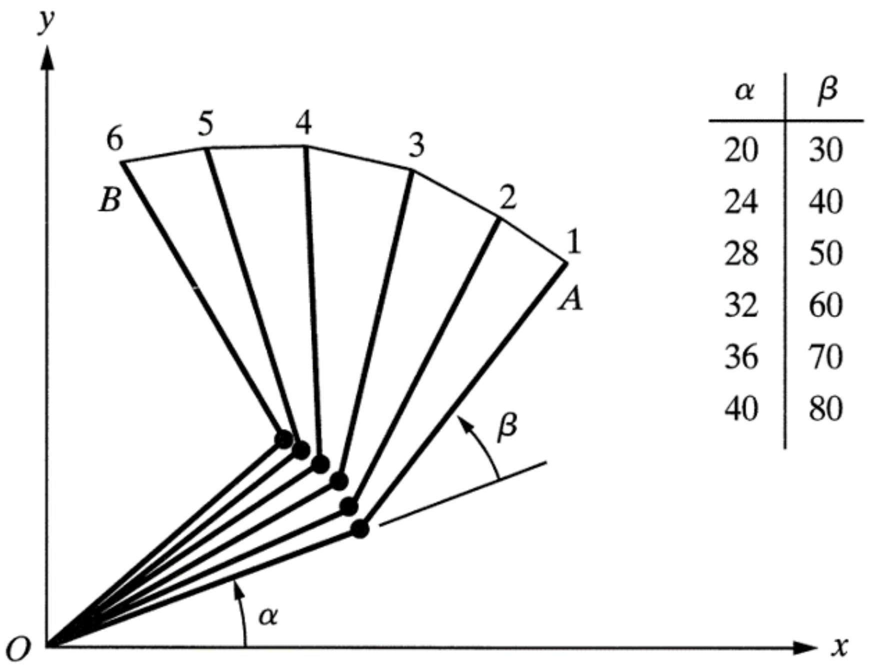
- Move the robot from A to B, to run both joints at their maximum angular velocities.
- After 2 [sec], the lower link will have finished its motion, while the upper link continues for another 3 [sec]
- The path is irregular and the distances traveled by the robot end point are not uniform.



α	β
20	30
30	40
40	50
40	60
40	70
40	80

Normalized trajectory

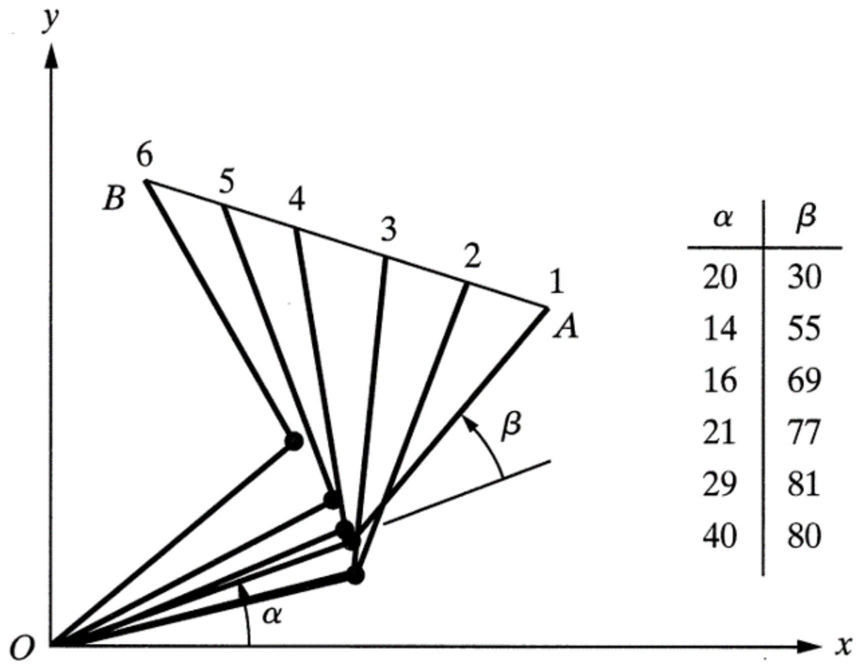
- Let us assume that the motions of both joints are normalized by a common factor such that the joint with smaller motion will move proportionally slower and the both joints will start and stop their motion simultaneously.
- Both joints move at different speeds, but move continuously together.
- The resulting trajectory will be different.



α	β
20	30
24	40
28	50
32	60
36	70
40	80

Straight line trajectory, version A

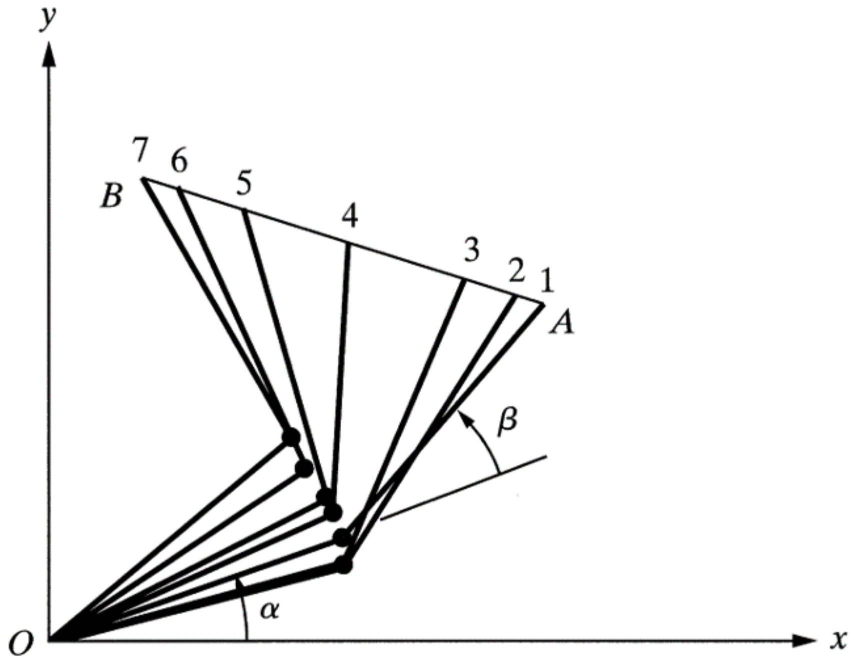
- Let us assume that the robot gripper follows a straight line between points A and B.
- The simplest way is to draw a line (interpolate) between A, B.
- Divide the line into five segments and solve for necessary angles α and β at each point.
- The joint angles do not change uniformly.



α	β
20	30
14	55
16	69
21	77
29	81
40	80

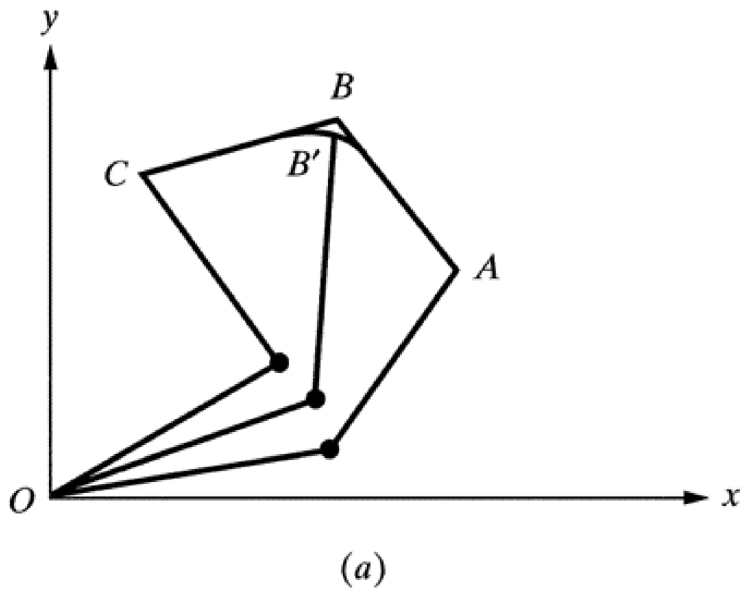
Straight line trajectory, version B

- Again interpolation between A, B by a straight line.
- The aim is to accelerate at the beginning and decelerate at the end.
- Divide the segments differently.
  - The arm moves at smaller segments as we speed up at the beginning;
  - Go at a constant cruising rate;
  - Decelerate with smaller segments as approaching Point B.



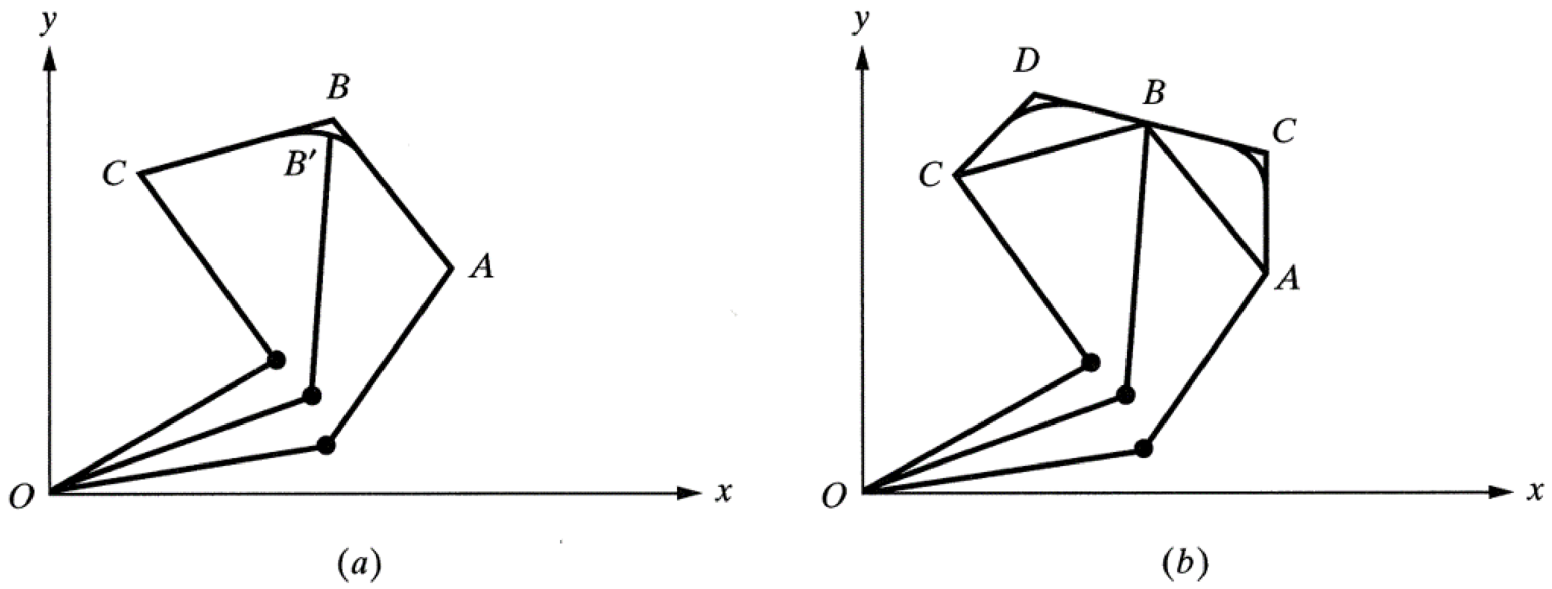
Continuous transition, version A

- Stop-and-go motion through the way-point list creates jerky motions with unnecessary stops.
- Stop-and-go motion through the way-point list creates jerky motions with unnecessary stops.
- How? Blend the two portions of the motion at Point B.

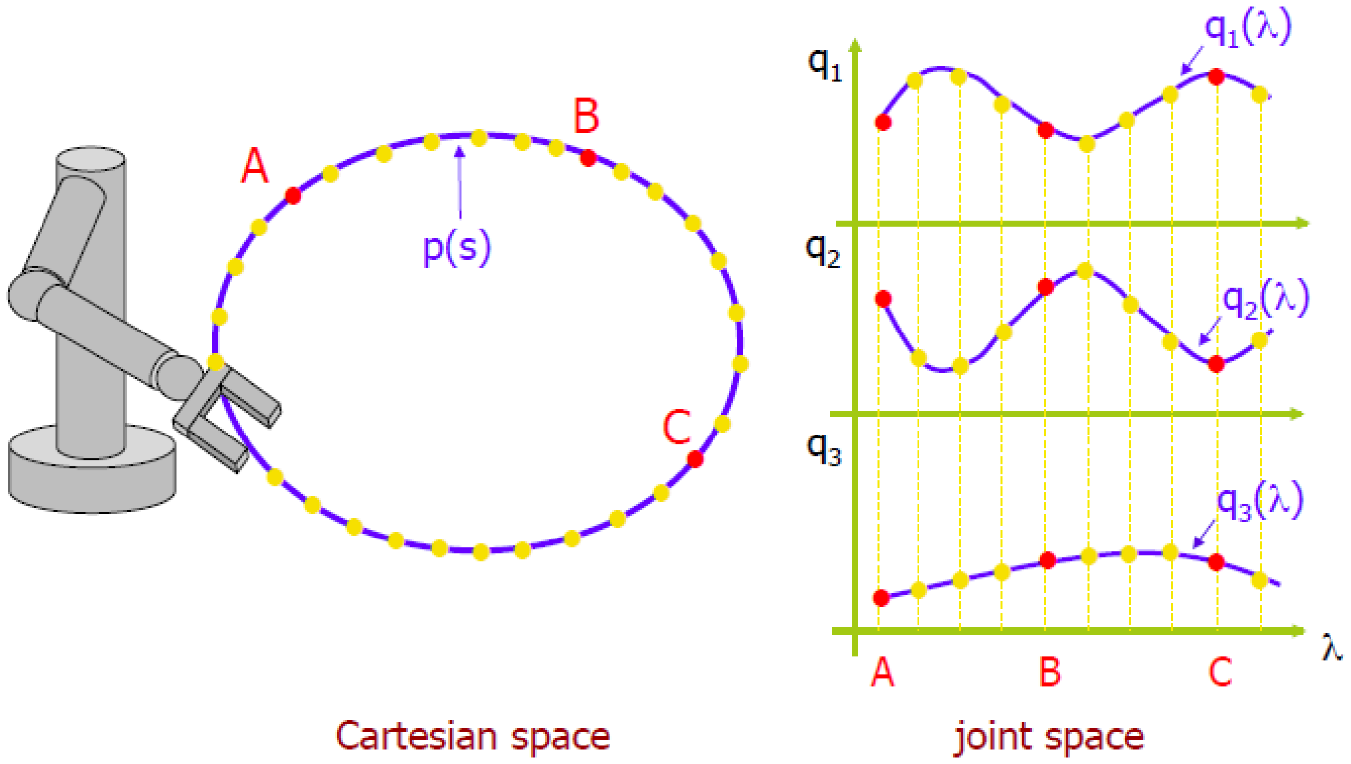


Continuous transition, version B

- Alternative scheme ensuring that the trajectory passes through control points.
- Two way-points D and E are picked such that Point B will fall on the straight-line section of the segment ensuring that the robot will pass through Point B.



Motivating example of trajectories in operational and joint spaces



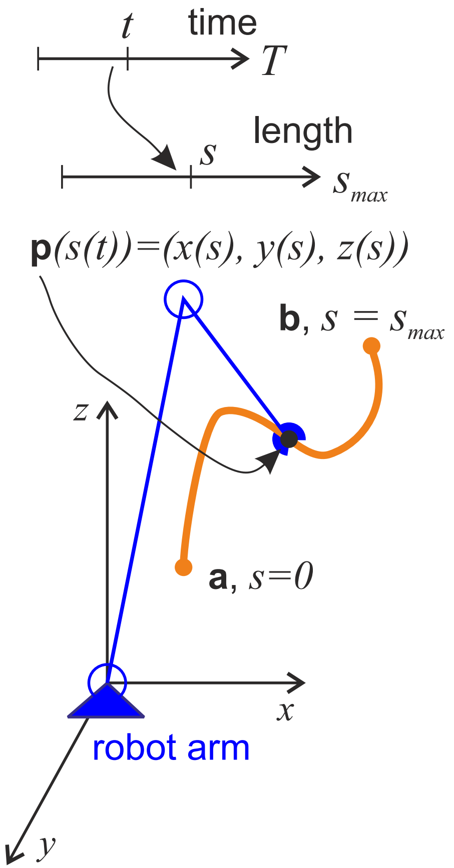
Courtesy for the image: Alessandro De Luca, Universita di Roma, La Sapienza.

Trajectory = path + timing law (1)

Having the path (approximated way-points), the trajectory is completed by a choice of a timing law.
Consider the geometric path in a parametric form:

- Operational (Cartesian) space: $\mathbf{p}(s) = (x(s), y(s), z(s))$, where the motion law is $s = s(t)$, $t \in [0, T]$.
- Joint space: $\mathbf{q}(\lambda) = (q_1(\lambda), q_2(\lambda), \ldots, q_n(\lambda))$, where $n =$ DOFs $\Rightarrow$ motion law $\lambda = \lambda(t)$.

If $s(t) = t$, the trajectory parametrization is the natural one given by the time.



Trajectory = path + timing law (2)

The timing law:

- is chosen based on task specifications (stop in a point, move at a constant velocity, etc.);
- It may consider optimality criteria (min transfer time, min energy, etc.);
- Constraints are imposed by actuator capabilities (e.g. max torque, max velocity) and/or by the task (e.g., the maximal allowed acceleration on a payload).

Space-time decomposition on parametrized path

E.g., in the operational (Cartesian) space:

$$\mathbf{p}(s) = (x(s), y(s), z(s)) \text{ with the motion law } s = s(t)$$

$$\text{velocity } \dot{\mathbf{p}}(t) = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s}\,\dot{s}(t), \qquad \text{acceleration } \ddot{\mathbf{p}}(t) = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s}\,\ddot{s}(t) + \frac{\mathrm{d}^2\mathbf{p}}{\mathrm{d}s^2}\,\dot{s}(t)$$

Polynomial functions of a degree n are employed usually for way-points approximation.

$$s(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

Trajectories classification

- Space of the definition: the operational (Cartesian) space or the joint space.
- Task type: point-to-point (PTP), multiple points (knots), continuous, concatenated.
- Path geometry: rectilinear, Dubins, polynomial, exponential, cycloid, ...
- Timing law: bang-bang in the acceleration, trapezoidal in the velocity, polynomial, ...
- Coordinated or independent:
  - Motion of all joints (or of all Cartesian components) start and ends at the same instants (say, $t = 0$ and $t = T$) $\Rightarrow$ the single timing law.
  - Motions are timed independently, e.g. according to requested displacement and robot capabilities. Such trajectory is performed in the joint space mostly.

Relevant characteristics of the trajectory

- Computational efficiency and memory space, e.g. store only coefficients of a polynomial function.
- Predictability vs. “wandering” out of the knots.
- Accuracy vs. the “overshot” on the final position.
- Flexibility allowing concatenation, over-fly, ...
- Continutity in space and in time.
  Continuity C^1 at least. Sometimes also up to C^2, i.e. up to jerk $= \frac{\mathrm{d}a}{\mathrm{d}t}$, where a is the acceleration.

Note: Continuity of the class C^k means that the time derivative up to the order k is smooth.

Trajectory planning

Usually, the user has to specify only a minimum amount of information about the trajectory, such as initial and final points, duration of the motion, maximum velocity, and so on.

- Work-space trajectories

  allow to consider directly possible constraints on the path (obstacles, path geometry, . . .) that are more difficult to take into consideration in the joint space (because of the non linear kinematics)

- Joint space trajectories

  are computationally simpler and allow to consider problems due to singular configurations, actuation redundancy, velocity/acceleration constraints.

Trajectory planning in the joint space

- The curve parametrization in time $q = q(t)$ or in joint space $q = q(\lambda)$, $\lambda = \lambda(t)$
- It is sufficient to work component-wise, (q_i in vector $\mathbf{q}$).
- An implicit definition of the trajectory is obtained by solving a problem with specified boundary conditions in a given class of functions.
- Typical classes of functions: polynomials (cubic, quintic, ...), (co)sinusoids, clothoids, etc.
- Imposed conditions
  - Passage through points $\Rightarrow$ interpolation.
  - Initial, final, intermediate velocity or geometric tangent to the path.
  - Initial, final, intermediate velocity or geometric curvature.
  - Continuity up to C^k.

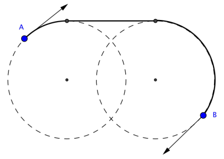
Many of the following methods and remarks can be applied directly also to Cartesian trajectory planning (and vice versa).

Dubins curves

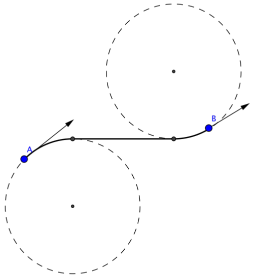
Lester Eli Dubins (1920–2010); result from 1957

The task is explained in 2D for simplicity; works in 3D as well.

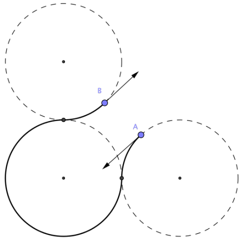
- Given: two points (start, target, both with prescribe tangent of the trajectory); Assumption: forward motion only.
- The optimization task is to find the shortest path between the start and the target when the maximum trajectory curvature is prescribed.
- L.E. Dubins proved that such path consists in a general case of at most (a) two maximum curvature two circular arcs and a straight line segment, or (b) three circular arcs.
- Let R - right turn; L - left turn; S - straight. The optimal trajectory will be one of six combinations RSR, RSL, LSR, LSL, RLR, LRL, see the figure.
- The methods are simple using algebraic equations.



RSR



RSL



LRL

A cubic polynomial

- Four boundary constraints:
  $q(0) = q_{\text{ini}}$; $q(T) = q_{\text{fin}}$; $\dot{q}(0) = v_{\text{ini}}$; $\dot{q}(T) = v_{\text{fin}}$.
- $\Delta q = q_{\text{fin}} - q_{\text{ini}}$; the curve parametrization $\tau = t/T$, $\tau \in [0, 1]$.
- A cubic polynomial $q(\tau) = q_{\text{ini}} + \Delta q \left(a\tau^3 + b\tau^2 + c\tau + d\right)$.
- A "doubly normalized" polynomial $q_N(\tau)$ such that
  - $q_N(0) = 0 \Leftrightarrow d = 0$.
  - $q_N = 1 \Leftrightarrow a + b + c = 1$.
  - $\dot{q}_N(0) = \left.\frac{\mathrm{d}q_N}{\mathrm{d}\tau}\right|_{\tau=0} = c = \frac{v_{\text{ini}}T}{\Delta q}$.
  - $\dot{q}_N(1) = \left.\frac{\mathrm{d}q_N}{\mathrm{d}\tau}\right|_{\tau=1} = 3a + 2b + c = \frac{v_{\text{fin}}T}{\Delta q}$.

Cubic polynomial, a special case, rest-to-rest

- The boundary constraints and the parametrization remains as above.
- A cubic polynomial as well: $q(\tau) = q_{\text{ini}} + \Delta q \left(a\tau^3 + b\tau^2 + c\tau + d \right)$.
- A polynomial $q_N(\tau)$ such that
  - $q_N(0) = 0 \Leftrightarrow d = 0$.
  - $\dot{q}_N(0) = 0 \Leftrightarrow c = 0$.
  - $q_N = 1 \Leftrightarrow a + b = 1$.
  - $\dot{q}_N(1) = 0 \Leftrightarrow 3a + 2b = 0$.

From previous two equations, $a = -2$, $b = 3$.

Quintic polynomial

- $q(\tau) = a\tau^5 + b\tau^4 + c\tau^3 + d\tau^2 + e\tau + f$, i.e., 6 coefficients,
- satisfying constraints, e.g., in the normalized time τ:
  $q(0) = q_0;\ q(1) = q_1;\ \dot{q}(0) = v_0 T; \dot{q}(1); \ddot{q}(0) = a_0 T^2;\ \ddot{q}(1) = a_1 T^2;$

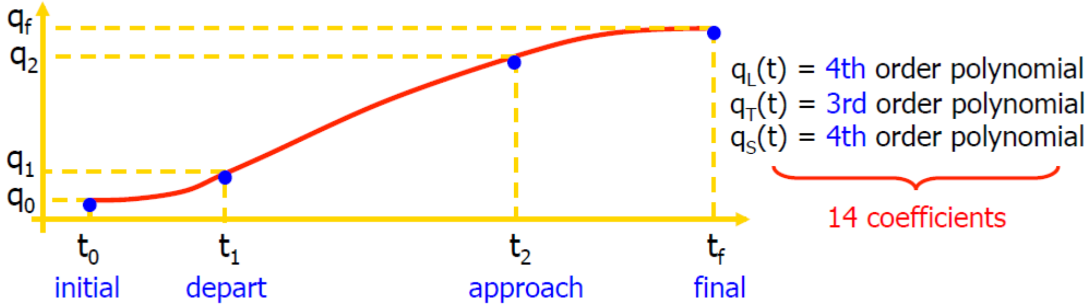
$$\begin{aligned} q(\tau) &= (1-\tau)^3 \left(q_0 + (3q_0 + v_0 T)\,\tau + (a_0 T^2 + 6 v_0 T + 12 q_0)\,\frac{\tau^2}{2} \right) \\ &= +\tau^3 \left(q_1 + (3q_1 - v_1 T)(1-\tau) + \frac{(a_1 T^2 - 6 v_1 T + 12 q_1)(1-\tau)^2}{2} \right) \end{aligned}$$

A special case, rest-to-rest:

- $v_0 = v_1 = a_0 = a_1 = 0.$
- $q(\tau) = q_0 + \Delta q(6\tau^5 - 15\tau^4 + 10\tau^3);\ \Delta q = q_1 - q_0.$

4-3-4 polynomials

Three phases in pick-and-place operations: Lift off, Travel, Set down.



Boundary constraints:

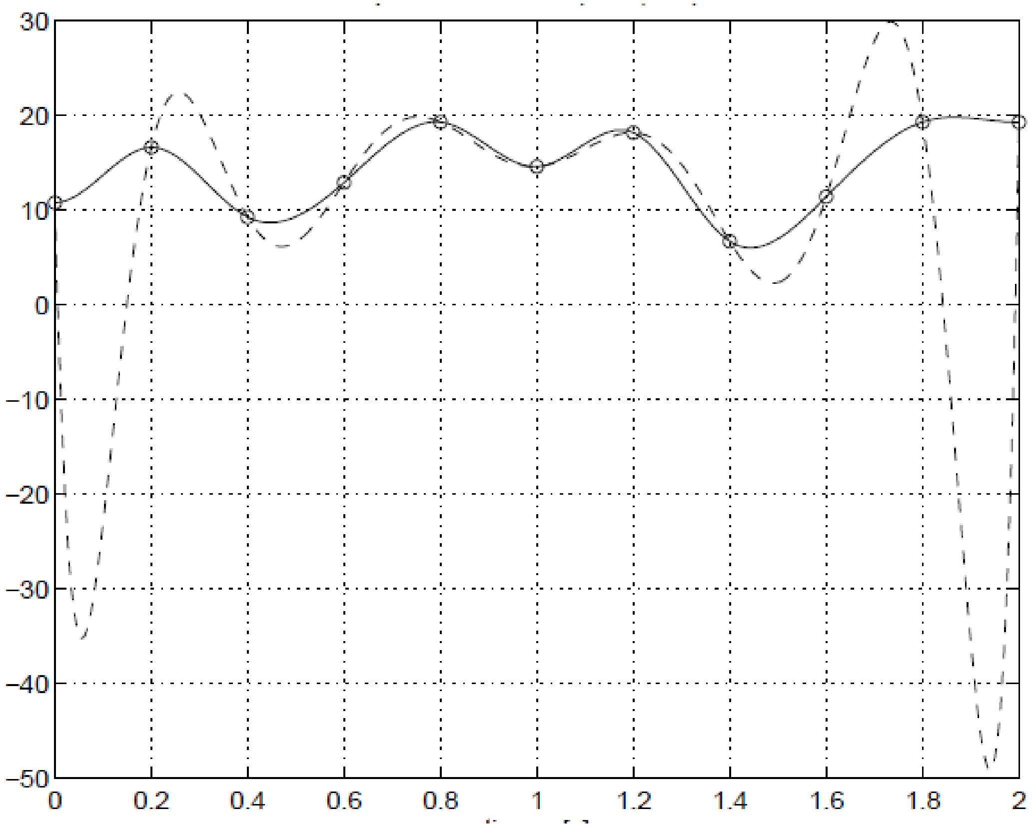
$$
\begin{aligned}
q(t_0) &= q_0, \quad q(t_1^-) = q(t_1^+) = q_1\,, \quad q(t_2^-) = q(t_2^+) = q_2\,, \; q(t_f) = q_f \\
\dot{q}(t_0) &= \dot{q}(t_f) = 0\,, \quad \ddot{q}(t_0) = \ddot{q}_f(t_f) = 0 \\
\dot{q}(t_0^-) &= \dot{q}(t_0^+)\,, \quad \ddot{q}(t_i^-) = \ddot{q}(t_i^+)\,, \quad i = 1, 2
\end{aligned}
$$

The first equation corresponds to six pasages; the second equation to four initial/final velocities/accelaretions; the third equation to four continuity constraints.

Higher-order polynomials

- Higher-order polynomials provide a suitable solution class for satisfying symmetric boundary conditions in a point-to-point motion that imposes zero values on higher-order derivatives.
  - The interpolating polynomial is always of the odd degree.
  - The coefficients of such (doubly normalized) polynomial are always integers, alternate in sign, sum up to unity, and are zero for all term up to the power $= \frac{(\text{degree}-1)}{2}$.
- In all other cases (e.g., for interpolating a large number N of points), the use of higher-order polynomials is not recommended.
  - N-th order polynomials have $N-1$ maximum and minimum points.
  - Oscilations arise out of the interpolation points (wandering).

Higher-order polynomials, oscillations

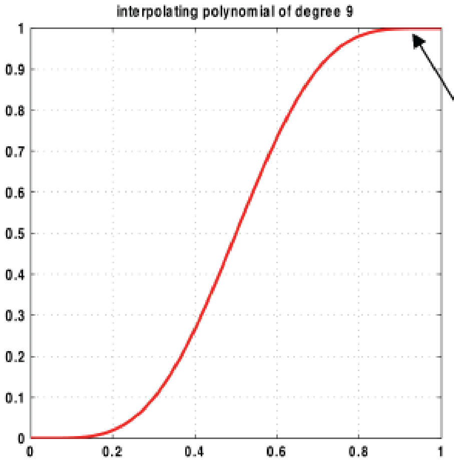


In general, given:

2 points	$\Longrightarrow$	unique line
3 points	$\Longrightarrow$	unique quadric
...		
n points	$\Longrightarrow$	unique polynomial with degree $n-1$

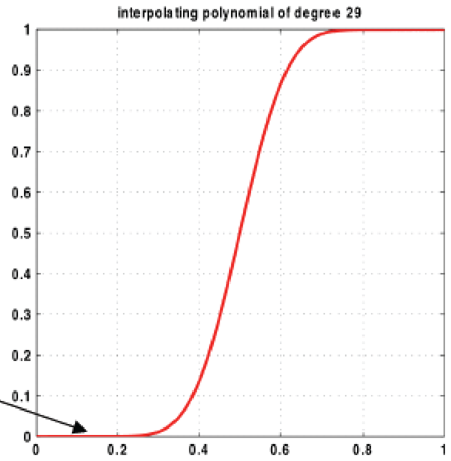
Numerical examples

9th degree



4 derivatives are zero

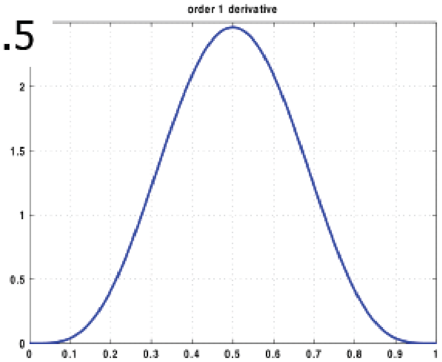
14 derivatives are zero!



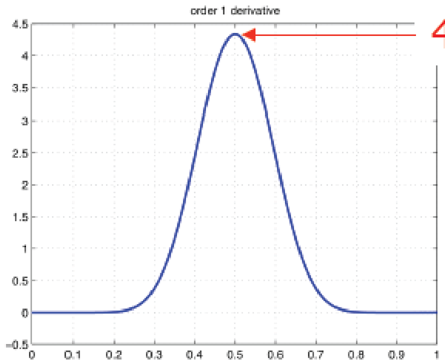
29th degree

no overshoot nor wandering

2.5



normalized velocity



4.5!!

velocity peaking at midpoint

Several polynomials instead of one high-degree polynomial

- Given N points, in order to avoid the problem of high 'oscillations' and troubles with the numerical precision avoid a single high-degree $N-1$ polynomial.
- Instead, use $N-1$ polynomials with lower degree p, $p < N-1$. Each polynomial interpolates a segment of the trajectory.
- Often $p = 3$ is chosen so that continuity of the velocity and acceleration is achieved.

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

- There are 4 coefficients for each polynomial, and thus it is necessary to compute $3(N-1)$ coefficients.

Spline

- The word "spline" refers to thin strip of wood or metal. At one time, curves were designed for ships or planes by mounting actual strips so that they went through a desired points but were free to move otherwise.
- Definition:

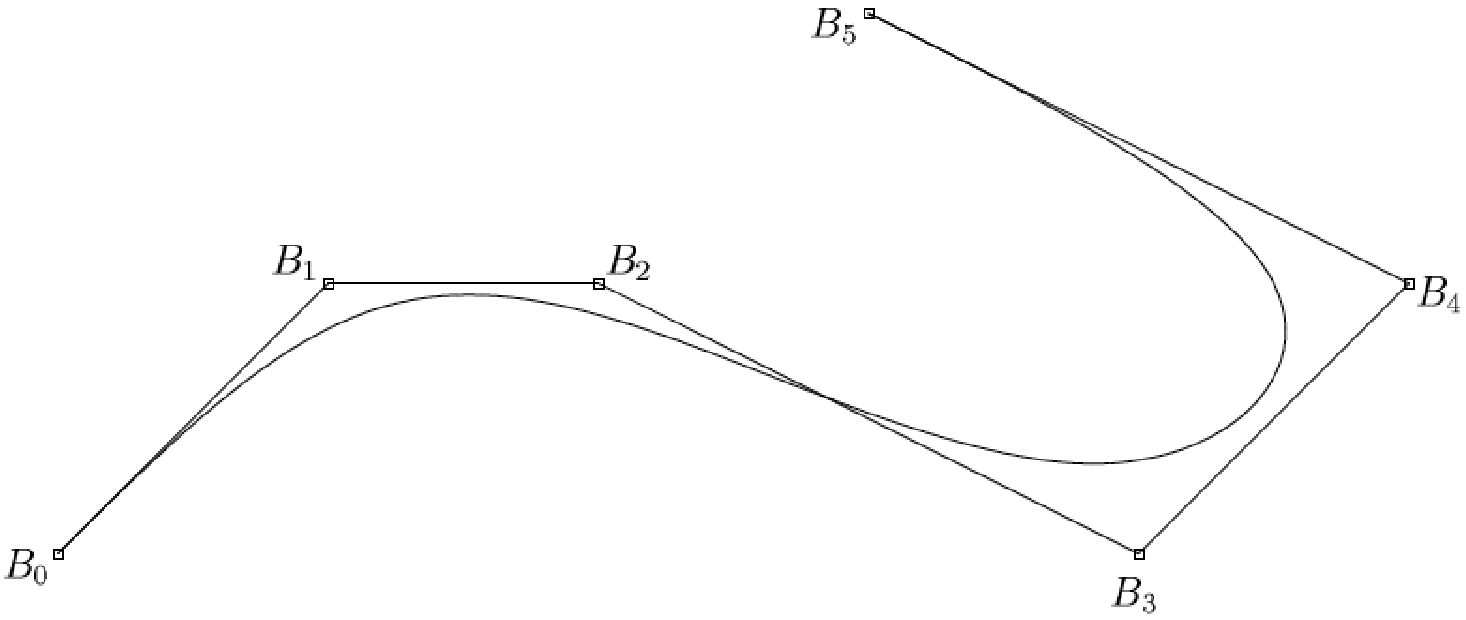
  A cubic spline curve is a piecewise cubic curve with continuous second derivation.
- Definition (a special case):

  A cubic spline curve is relaxed if its second derivative is zero at each endpoint.
- An easy way of making a controlled-design curve with many control points is to use B-spline curves.

B-spline, construction by hand (1)

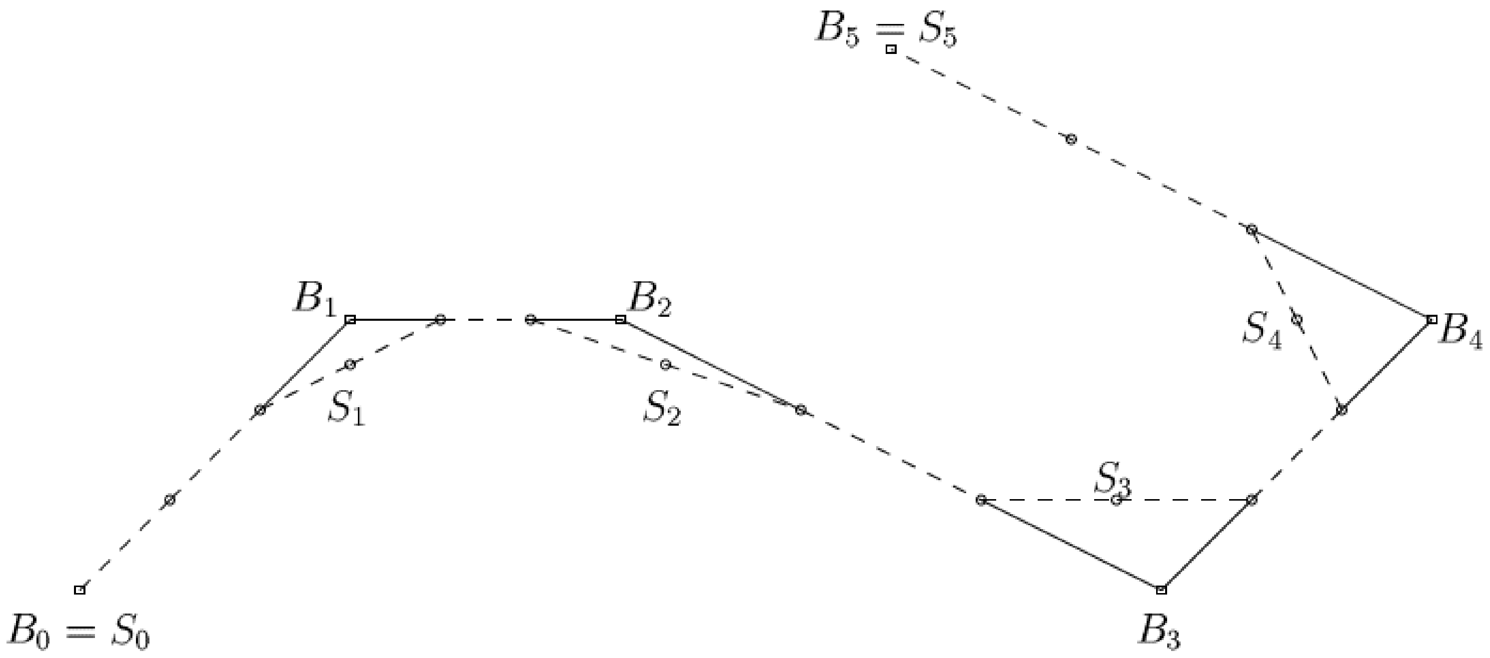
We start the description with relaxed uniform B-spline curves for simplicity.

Assume starting by specifying of a control polygon of points $B_0, B_1, \ldots, B_n$.



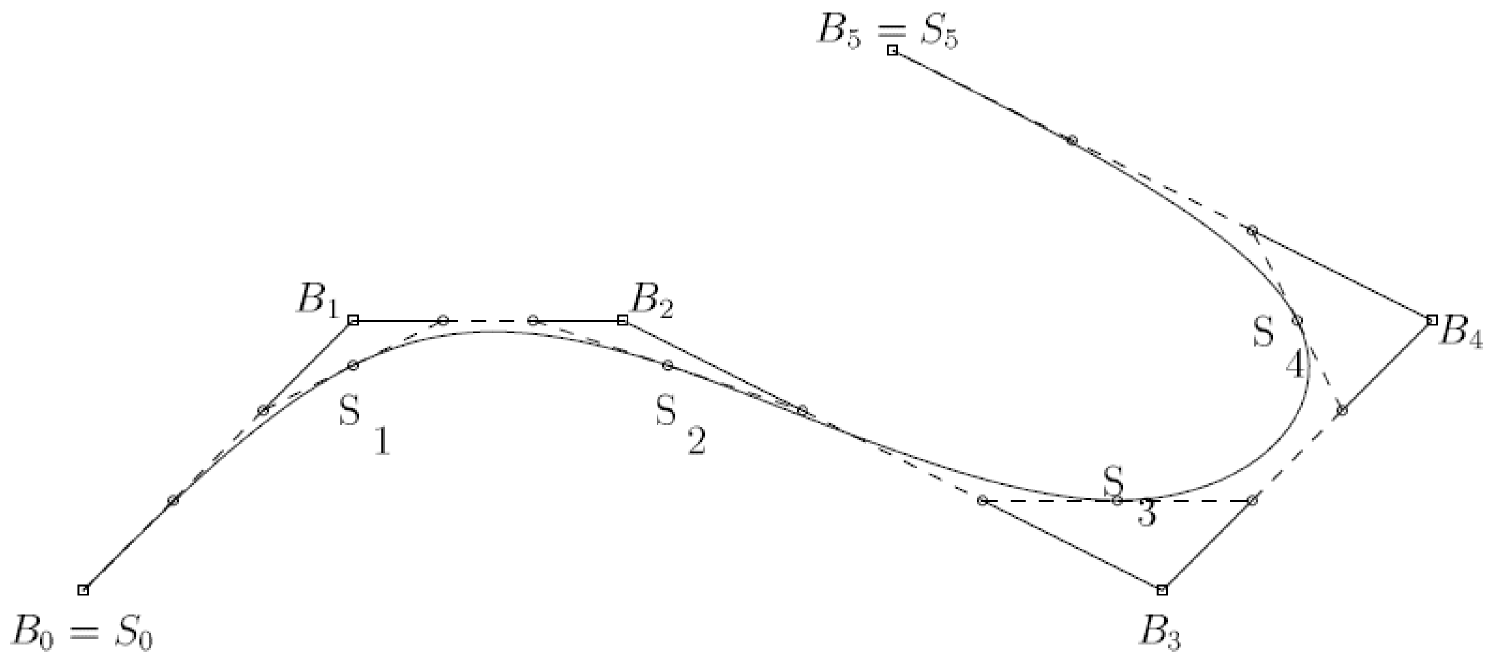
B-spline, construction by hand (2)

The B-spline construction method if done by hand: Divide each leg of a control polyton in thirds by marking two “division” points. At each B_i except the first and last, draw a line segments between the two nearest “division” points. Call the midpoint S_i. Then you make an A-frame with B_i at the apex. Let $S_0 = B_0$ and $S_n = B_n$ for completeness.



B-spline, construction by hand (3)

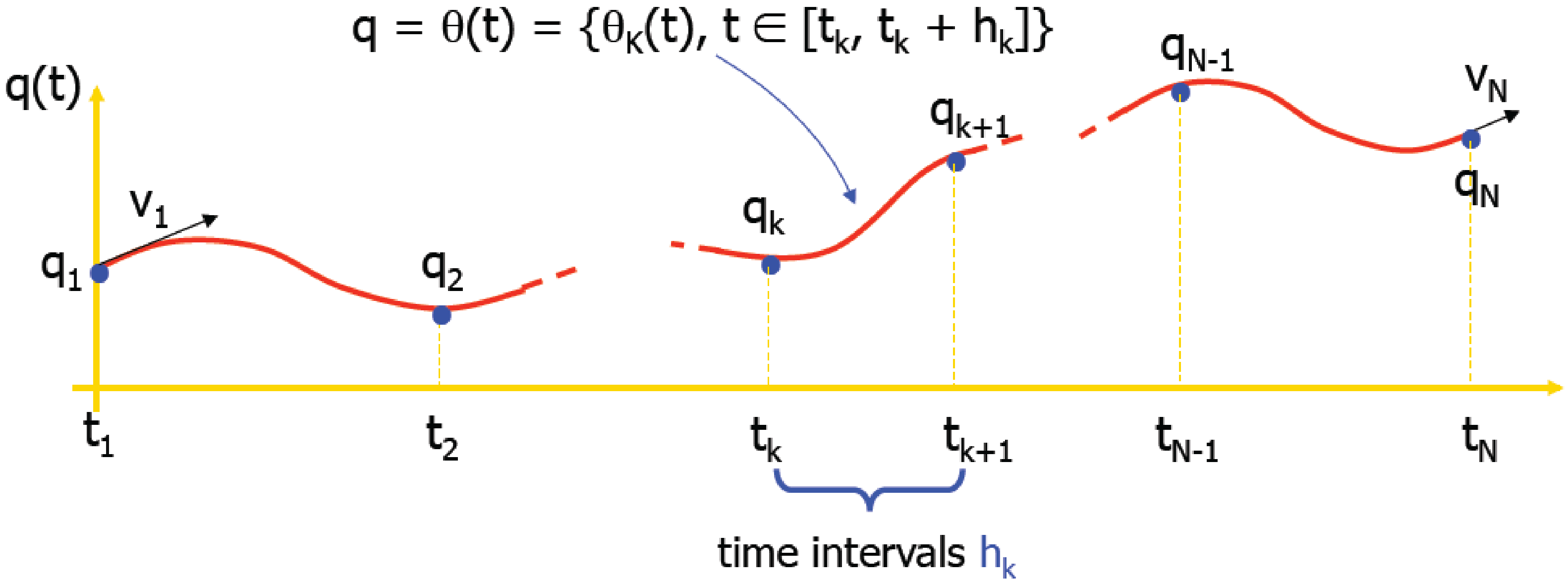
Sketch a cubic (Bézier) curve from each point S_i to the next using as Bézier control points the four points S_i, two "division" points, and S_{i+1}.



Interpolation using splines

- Task formulation: Interpolate N knots along the desired curve, keep C^2 continuity, i.e. continuity up to the second derivative.
- Solution: Approximate by splines.
  Splines are $N-1$ cubic polynomials concatenated to pass through knots (also control points) and being continuous in velocity and acceleration in the $N-2$ internal knots.
- $4\,(N-1)$ coefficients.
- $4\,(N-1)-2$ conditions, more specifically
  - $2\,(N-1)$ passages, pro each cubic in the two knots at its ends.
  - $N-2$ continuities for velocities at internal knots.
  - $N-2$ continuities for accelerations at internal knots.
- Two free parameters are still left over. These parameters can be used to, e.g., assign the initial and final velocities v_1, v_N.
- We present curves in terms of time t. It is similar for space λ.

Building a cubic spline



- The polynomial $\Theta_K(\tau) = a_{k0} + a_{k1}\tau + a_{k2}\tau^2 + a_{k3}\tau^3$; $\tau \in [0, h_k]$, $\tau = t - t_k$, $k = 1, \ldots, N-1$.
- Continuity conditions for velocity and acceleration: $\dot{\Theta}_k(h_k) = \dot{\Theta}_{k+1}(0)$; $\ddot{\Theta}_k(h_k) = \ddot{\Theta}_{k+1}(0)$; $k = 1, \ldots, N-2$.

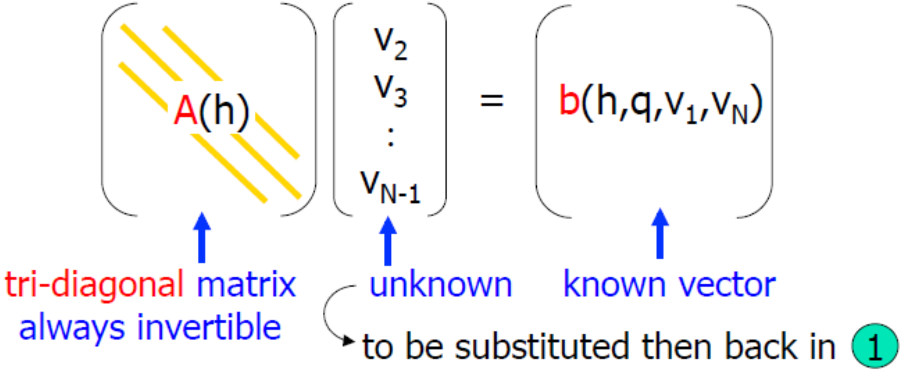
An efficient algorithm

1. If all velocities v_k at internal knots were known then each cubic in the spline would be uniquely determined by

$$\begin{aligned} \Theta_k(0) &= q_k = a_{k0} \\ \dot{\Theta}(0) &= v_k = a_{k1} \end{aligned} \qquad \begin{bmatrix} h_{k2} & h_{k3} \\ 2\,h_k & 3\,h_{k2} \end{bmatrix} \begin{bmatrix} a_{k2} \\ a_{k3} \end{bmatrix} = \begin{bmatrix} q_{k+1} - q_k - v_k\,h_k \\ v_{k+1} - v_k \end{bmatrix}$$

2. Impose $N-2$ acceleration continuity constraints
$\ddot{\Theta}_k(h_k) = 2a_{k2} + 6a_{k3}h_k = \ddot{\Theta}_{k+1}(0) = 2\,a_{k+1,2}$

3. Expressing the coefficients a_{k2}, a_{k3}, $a_{k+1,2}$ in terms of still unknown knot velocities, see step 1, yields a linear system of equations, which are always solvable



Structure of $A(h)$

$$
\begin{pmatrix}
2(h_1+h_2) & h_1 & & & \\
h_3 & 2(h_2+h_3) & h_2 & & \\
 & & \ddots & & \\
 & & h_{N-2} & 2(h_{N-3}+h_{N-2}) & h_{N-3} \\
 & & & h_{N-1} & 2(h_{N-2}+h_{N-1})
\end{pmatrix}
$$

diagonally dominant matrix (for $h_k > 0$)
[the same matrix for all joints]

Structure of $b(h, q, v_1, v_N)$

$$\begin{pmatrix} \frac{3}{h_1 h_2}[h_1^2(q_3 - q_2) + h_2^2(q_2 - q_1)] - h_2 v_1 \\ \frac{3}{h_2 h_3}[h_2^2(q_4 - q_3) + h_3^2(q_3 - q_2)] \\ \vdots \\ \frac{3}{h_{N-3} h_{N-2}}[h_{N-3}^2(q_{N-1} - q_{N-2}) + h_{N-2}^2(q_{N-2} - q_{N-3})] \\ \frac{3}{h_{N-2} h_{N-1}}[h_{N-2}^2(q_N - q_{N-1}) + h_{N-1}^2(q_{N-1} - q_{N-2})] - h_{N-2} v_N \end{pmatrix}$$

Properties of splines

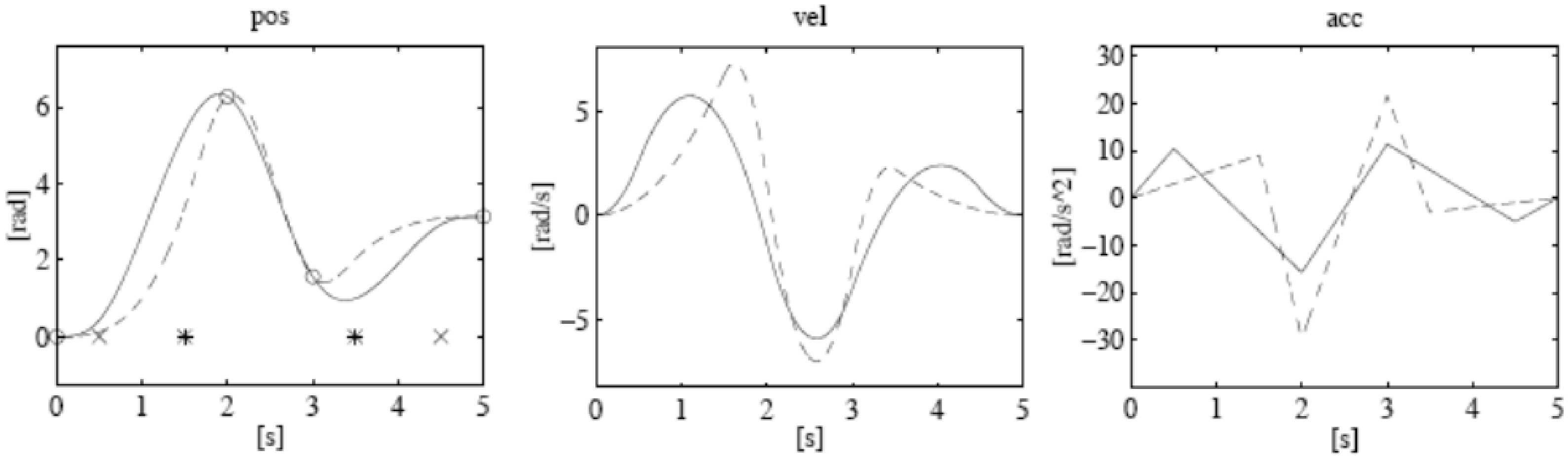
- The spline is the solution with the minimum curvature among all interpolating functions having continuous derivatives up to the second one.
- A spline is uniquely determined from the data $q_1, \ldots, q_n$, $h_1, \ldots, h_{N-1}$, $v_1, \ldots, v_n$.
- The total transfer time is $T = \sum_{k=1}^{K} h_k = t_N - t_1$.
- The time intervals h_k can be chosen to minimize T (linear objective function) under (nonlinear) bounds on velocity and acceleration in $[0, T]$.
- For cyclic taks ($q_1 = q_N$), it is preferable to impose simply the continuity of velocity and acceleration at $t_1 = t_N$ as the "squaring" conditions
  - in fact, even choosing $v_1 = v_N$ does not guarantee the acceleration continuities;
  - in this way, the first $=$ last knot will be handled as all other internal knots
- When initial and final accelerations are also assigned, the spline construction can be suitably modified.

A modification
handling assigned initial and final accelerations

- Two more parameters are needed in order to impose also the initial acceleration α_1 and final acceleration α_N.
- Two "fictious knots" are inserted in the first and last original intervals, increasing the number of cubic polynomials from $N-1$ to $N+1$.
- In these two knots only continuity conditions on position, velocity and acceleration are imposed $\Rightarrow$ two free parameters are left over (one in the first cubic and one in the last cubic), which are used to satisfy the boundary constraints on acceleration.
- Depending on the (time) placement of the two additional knots, the resulting spline changes.

A numerical example

- $N = 4$ knots (3 cubic polynomials)
  - joint values $q_1 = 0$, $q_2 = 2\pi$, $q_3 = \pi/2$, $q_4 = \pi$
  - at $t_1 = 0$, $t_2 = 2$, $t_3 = 3$, $t_4 = 5$ (thus, $h_1 = 2$, $h_2 = 1$, $h_3 = 2$)
  - boundary velocities $v_1 = v_4 = 0$
- 2 added knots to impose accelerations at both ends (5 cubic polynomials)
  - boundary accelerations $\alpha_1 = \alpha_4 = 0$
  - two placements: at $t_1' = 0.5$ and $t_4' = 4.5$ ($\times$), or $t_1'' = 1.5$ and $t_4'' = 3.5$ ($*$)



——— = placement′ ---------- = placement″

References

- B. Siciliano et al. Robotics (Modeling, planning and control), Springer, Berlin, 2009, chapter 4: Trajectory planning, pages 161-189.