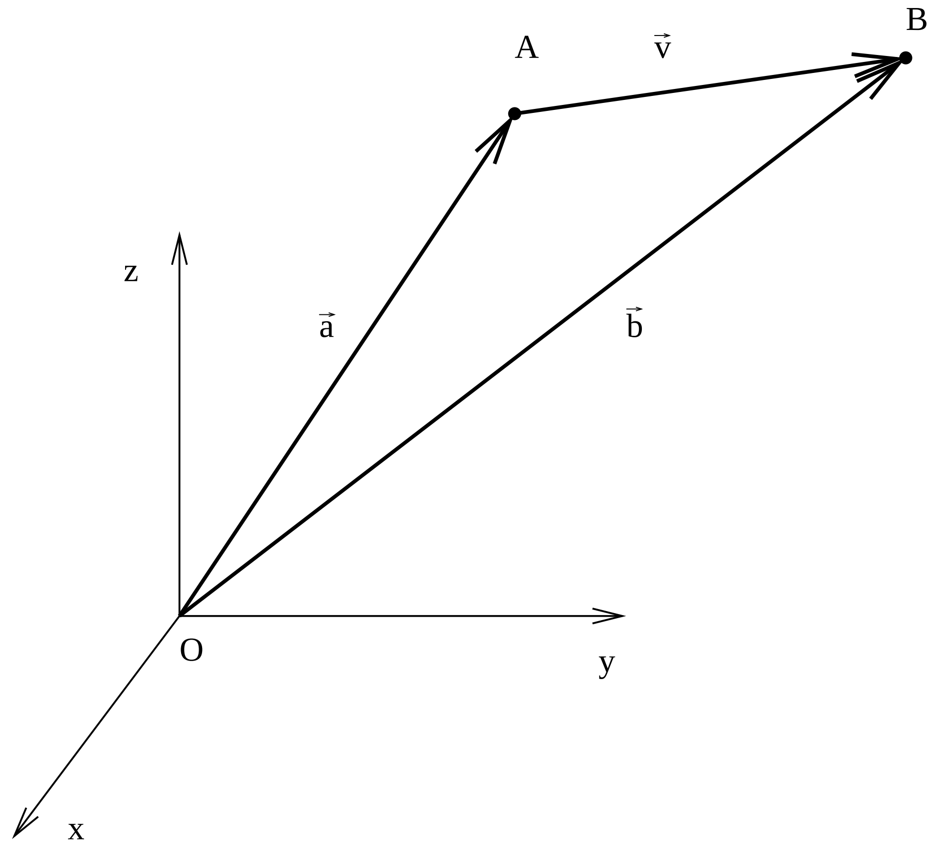
Points, Vectors, Geometry, Algebra



Points and vectors in geometry and algebra

The notion *point* in geometry is considered here as well known and as its definition is not mathematically simple, I will not define it here. Let us emphasize that it should be considered as "place in the space", which is independent on any numbers.

The term *vector* in geometry we can define as ordered pair of points, which are called initial and terminal point of the vector. We denote vector as $\vec{AB}$.

We can use the phrases in geometry as "point A is intersection of circle k_1 with the center in point B and containing point C and line passing through points D and E".

The problems in robotics are strictly speaking geometrical problems (at least in kinematics). We need to represent the geometry of robots in computer to be able to control robots. Unfortunately the digital computer cannot handle geometry directly.

Mathematical theory of linear spaces allows us to handle mathematical objects like ordered n-tuple of numbers. It is possible to show that structure preserving mapping (isomorphism) between geometrical and algebraic objects exists for Euclidean geometry and algebraic linear spaces of the finite dimension. To introduce such isomorphism, we need to introduce coordinate system. Geometrical point in space could be then made identical to the ordered triple of real numbers. This triple is usually called vector, but this time being the algebraic term. The real numbers in the triple mean coordinates of geometrical point in the coordinate system. The radius vector in geometry could define the position of the point in the space. The initial point of the radius vector is in the origin of the coordinate system, the final point of the radius vector is the represented geometrical point.

(Algebraic) linear space thus makes universal model of geometrical space and using the linear space we can represent all properties of geometrical space and vice versa.

Considering the geometrical terms as primary can be justified by the historical fact, that geometry was cultivated by mathematicians 2000 years without introducing coordinates. The existence of isomorphism demonstrates that both descriptions of the reality are (mathematically) equivalent and we can switch between them arbitrarily. We can define between points and vectors in geometry several operators:

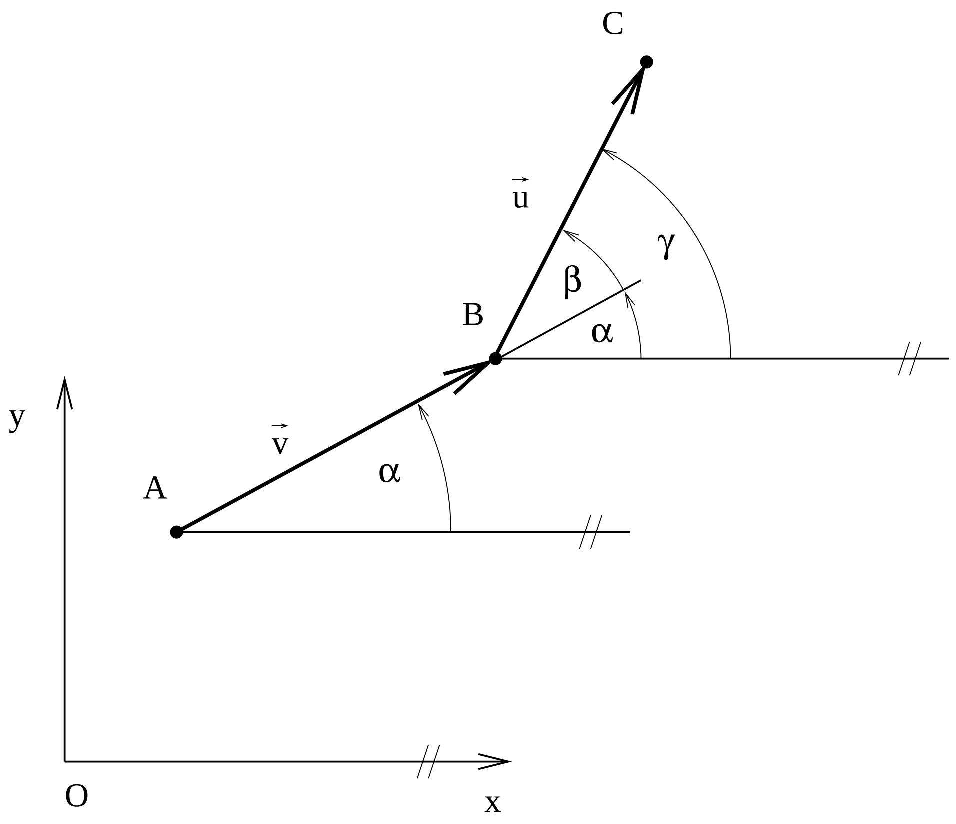
- two points define vector $\vec{v} = \vec{AB}$,
- point and vector define point $A + \vec{v} = B$
- vectors could be added and subtracted $\vec{b} = \vec{a} + \vec{v}$

These operations hold without introducing coordinate system.

The same operations could be introduced in algebra between ordered triples (n-tuples) representing points and vectors. The formulas remain the same, the meaning of the objects in formulas differs. Algebraic objects in formulas correctly represent geometrical objects if the coordinates of the geometrical objects are in the same coordinate system. Further holds that coordinate system can be chosen arbitrarily. Form of the algebraic description will be for any choice of coordinate system same, the numbers in equations will depend on the choice of the coordinage system.

Situation with several coordinate systems will be described later.

Technical intermezzo



The common approach for solving problems of determining parameters of triangle is to use e.g. Law of Cosine or Law of Sine. Such laws, sum of inner angles and so on, are difficult to apply in analytical geometry. The problem is that found solutions should be interpreted for different quadrants, new solutions shall be constructed or found solutions tested for satisfying input conditions. The reason is, that the orientation of angles cannot be determined from such computations and interpretation of results is difficult.

I recommend to avoid such approaches as much as possible and use the following reliable tools.

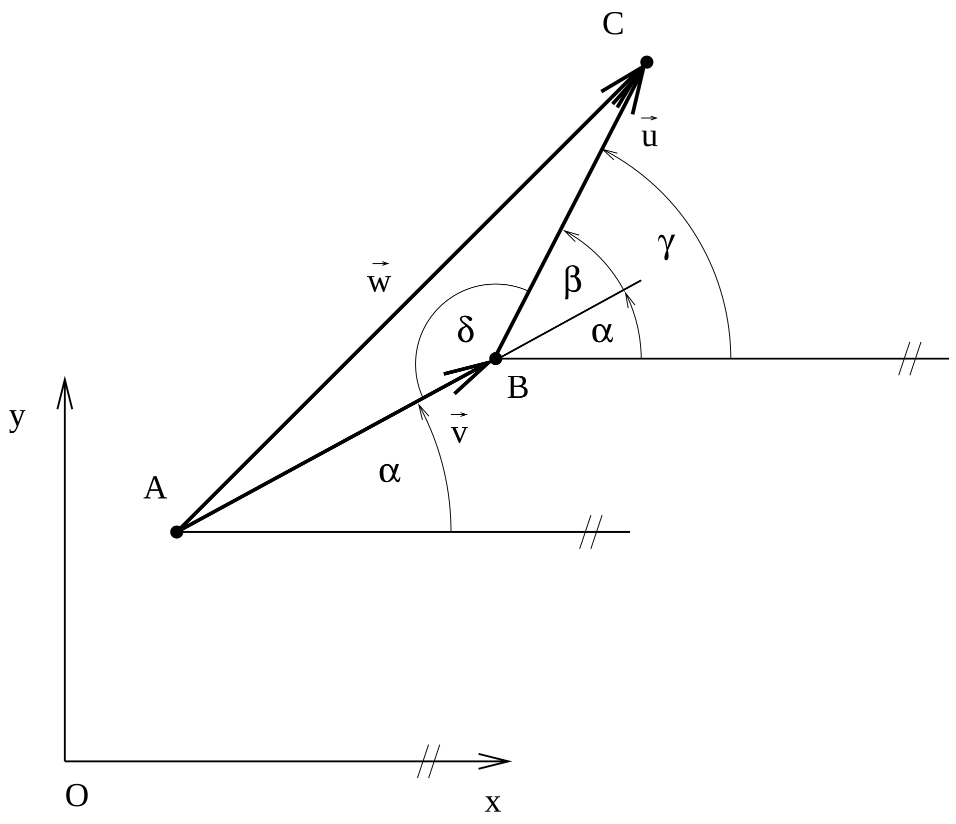
- Try to calculate coordinates of corners instead of length of sides or size of angles.
- Use $\phi = \mathrm{atan2}(y, x)$ for determining angle as much as possible. Avoid arccos as much as possible.
- When calculating angles and coordinates, strictly use oriented versions of them. They can be then sumed and subtracted without analysis of particular situations.
- If you need analytical equation of line, use the form $ax + by + c = 0$, which unlike $y = kx + q$ works in all quadrants.

See the calculation of the angle between x axis and $\vec{AB}$ vector (see figure above). The oriented, four quadrant angle α could be calculated as $\alpha = \mathrm{atan2}(B_y - A_y, B_x - A_x)$.

The safest and easiest way, how to calculate the oriented angle β between vectors $\vec{v}$ and $\vec{u}$ is the following:

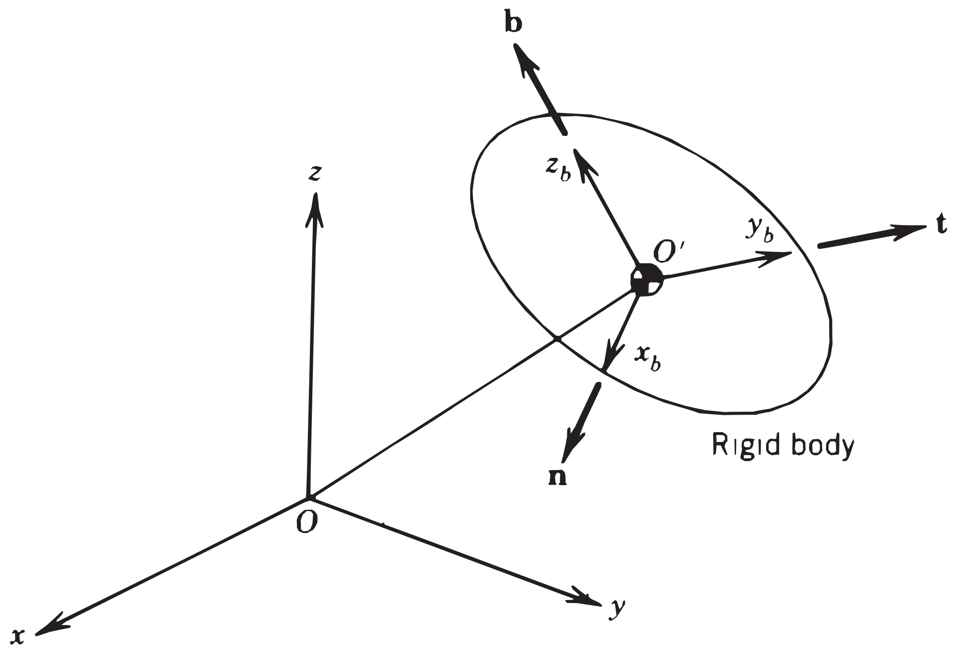
$$\alpha = \mathrm{atan2}(B_y - A_y, B_x - A_x), \quad (1)$$
$$\gamma = \mathrm{atan2}(C_y - B_y, C_x - B_x), \quad (2)$$
$$\beta = \gamma - \alpha. \quad (3)$$

Technical intermezzo II



Let us have given the coordinates of points A and C and lengths of vectors $\vec{v}$ and $\vec{u}$. The safest way how to determine the point B and associated angles, is not to fight via Law of Cosine with the angle δ but to calculate the intersection of circles with the centers in A and C and appropriate radii. This gives two solutions for point B. The angles could be then determined by the above equations.

Body in the coordinate system and its motion



The rigid body in a plane has 3 degree of freedom.
The same body in 3D space has 6 degree of freedom:
Questions:
How many DOF has a rubber tape?

Rigid body – in this course we always consider links and manipulated objects as rigid bodies. We can attach the coordinate system to the rigid body and position of the individual points on the rigid body is then unique and supposed to be known apriori, e.g. from CAD drawings of the body.

Actual body position in time could be described by the position of the attached coordinate system in other, nonmoving, "world" coordinate system.

Motion of the body in time could be described as a actual position of the body as a function of time.

Relative position of two coordinates system can always be decomposed into the translation and rotation.

Let the coordinate system of a base is $O - xyz$. The coordinate system of a body is $O' - x^b y^b z^b$. The description of a coordinate system $O' - x^b y^b z^b$ in coordinate system of a base is:

$$\vec{OO'} = \mathbf{x}_o = \begin{pmatrix} x_o \\ y_o \\ z_o \end{pmatrix} (\mathbf{n}, \mathbf{t}, \mathbf{b})\,.$$

Let us form a matrix $\mathbf{R} = (\mathbf{n}, \mathbf{t}, \mathbf{b})$, $\mathbf{n}$, $\mathbf{t}$, $\mathbf{b}$ are unit and orthogonal vectors, then a matrix $\mathbf{R}$ is ortonormal, that is $\mathbf{R}^{-1} = \mathbf{R}^T$.

Description of Body Position

Point in 3D - described by three coordinates.

Rigid body in 3D - described by 6 coordinates:

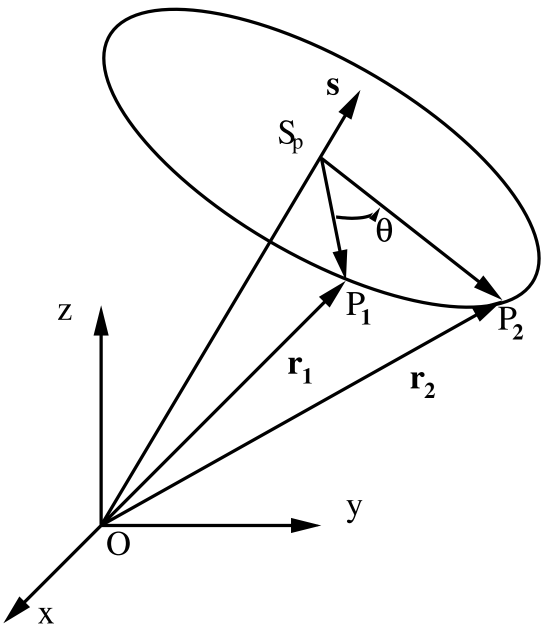
- 3 coordinates of reference point $\mathbf{t}_0^0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$,
- orientation could be described e.g. by:
  - coordinates of vectors attached to the body $(\mathbf{n}, \mathbf{t}, \mathbf{b})$,
  - Euler angles (ϕ, θ, ψ),
  - rotational matrix $\mathbf{R}$,
  - axis – angle,
  - quaternions,
  - rotation vector.

Coordinates of reference point and rotation matrix could be combined into transformation matrix.

Rotation matrix see Eric W. Weisstein. "Rotation Matrix." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/RotationMatrix.html

For conversion among different descriptions see http://www.euclideanspace.com/maths/geometry /rotations/conversions/index.htm Pay attention to the definitions used so you do mix different definitions from different sources.

Axis–Angle, Quaternions, Rotation Vector



Euler's rotation theorem states than any rotation in 3–D could be represented as a single rotation around a certain axis. We can describe this axis as $\mathbf{s}$ and angle of the rotation as θ. This pair $(\mathbf{s}, \theta)$ could represent rotation and is called axis–angle. Quaternions can be expressed as:

$$\mathbf{q} = (\cos(\theta/2), \sin(\theta/2)\mathbf{s}^T) =$$
$$(\cos(\theta/2), \sin(\theta/2)s_x, \sin(\theta/2)s_y, \sin(\theta/2)s_z)$$

Rotation vector uses the fact, that vector $\mathbf{s}$ is normalized and has only 2 DOF, so one can express rotation using three numbers $\mathbf{v} = (\theta\mathbf{s})$. Rodrigues' rotation formula:

$$\mathbf{r}_2 = \mathbf{r}_1 \cos\theta + (\mathbf{s} \times \mathbf{r}_1)\sin\theta + \mathbf{s}(\mathbf{s} \cdot \mathbf{r}_1)(1 - \cos\theta)$$

$$R = I\cos\theta + [\mathbf{s}]_x \sin\theta + \mathbf{s}\mathbf{s}^T(1 - \cos\theta)$$

Quaternions have various properties, among them:

- all four components are equal "important", unlike Euler angles,
- $\mathbf{q}$ and $-\mathbf{q}$ describe same rotation,
- it is relatively easy to interpolate the rotation in quaternion parametrisation.

The task to rotate smoothly (e.g. manipulate in robotics or display in computer graphics) could be easily achieved using quaternions. The method is called spherical linear interpolation (SLERP). We interpolate from $\mathbf{q}_1$ to $\mathbf{q}_2$ getting $\mathbf{q}$ and having interpolating parameter t, the angle α is the amount of rotation needed (angle between quaternions interpreted as four dimensional vectors is half of it, absolute value assures the shorter rotation):

$$\mathbf{q} = (\mathbf{q}_2 \cdot \mathbf{q}_1^{-1})^t \mathbf{q}_1 \,, \quad (4)$$

$$\mathbf{q} = \frac{\sin((1-t)\alpha)}{\sin(\alpha)}\mathbf{q}_1 + \frac{\sin(t\alpha)}{\sin(\alpha)}\mathbf{q}_2 \,, \quad (5)$$

$$\cos(\alpha/2) = \|\mathbf{q}_1 \cdot \mathbf{q}_2\| \,, \quad (6)$$

$$t \in \, <0, 1> \,. \quad (7)$$

For details about quaternions see Eric W. Weisstein. "Quaternion." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Quaternion.html

Interpolation in axis-angle system is rotation around given axis, where angle changes linearly from 0 to θ.

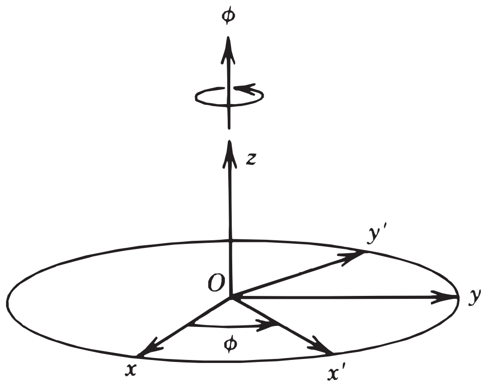
The rotation vector is nonredundant and has good topology, so it is heavily used e.g. in computer vision for rotation estimation. Good topology means here that rotation vectors which are closed to each other (their difference is small) represent rotations which differ only little. This hols also near zero rotation vector.

Rodrigues' rotation formula allow to calculate vector r_2 or equivalently point P_2 when rotating around axis s by an angle θ. The modified Rodriques' formula allows to calculate easily rotation matrix from axis–angle representation. The inverse transformation is:

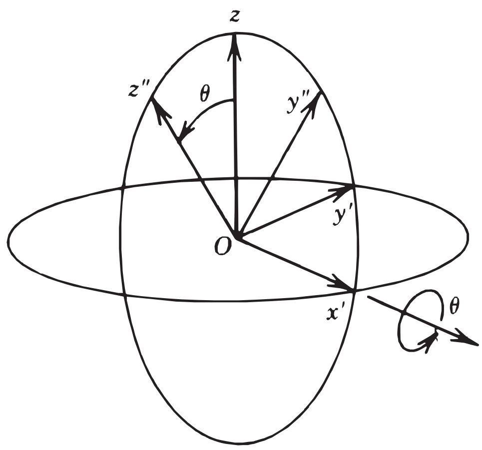
$$\theta = \arccos\left(\frac{\mathrm{trace}(\mathbf{R} - 1)}{2}\right) \quad (8)$$

$$\mathbf{s} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{3,2} - r_{2,3} \\ r_{1,3} - r_{3,1} \\ r_{2,1} - r_{1,2} \end{bmatrix} \quad (9)$$

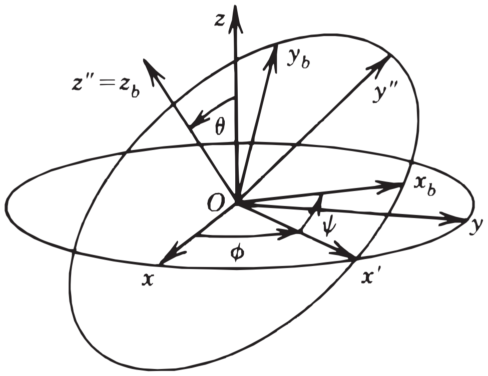
Definition of Euler angles



1 – precession



2 – nutation



3 – rotation

Euler angles

The matrix $\mathbf{R}$ has nine coeficients, but its dimension is only three, the bounding condition is unit size and orthogonality of vectors $\mathbf{n}$, $\mathbf{t}$, $\mathbf{b}$:

$$\mathbf{n}^T\mathbf{t} = 0 \quad \mathbf{t}^T\mathbf{b} = 0 \quad \mathbf{b}^T\mathbf{n} = 0$$
$$|\mathbf{n}| = 1 \quad |\mathbf{t}| = 1 \quad |\mathbf{b}| = 1$$

The matrix $\mathbf{R}$ can be contructed easily using Euler angles

1. Rotate the coordinate system $O - xyz$ around the axis z by the angle ϕ. We will get $O - x'y'z$.
2. Rotate the coordinate system $O - x'y'z$ around the axis x' by the angle θ. We will get $O - x'y''z''$.
3. Rotate the coordinate system $O - x'y''z''$ around the axis z'' by the angle ψ. We will get $O - x^b y^b z^b$.

$$\mathbf{R} = \mathbf{R}_z(\phi)\mathbf{R}_{x'}(\theta)\mathbf{R}_{z''}(\psi)$$

$$\mathbf{R}_z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (10)$$

$$\mathbf{R}_{x'}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad (11)$$

$$\mathbf{R}_{z''}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (12)$$

$$\mathbf{R} = \begin{bmatrix} \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi & -\cos\theta\cos\psi\sin\phi - \cos\phi\sin\psi & \sin\phi\sin\theta \\ \cos\psi\sin\phi + \cos\phi\cos\theta\sin\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \cos\psi\sin\theta & \cos\theta \end{bmatrix} \quad (13)$$

Euler angles define uniquely the rotation, the same rotation can be reached by different triples of angles. Other definitions like roll–pitch–yaw define similar angles with similar properties but with different equations. If the matrix $\mathbf{R}$ is given, Euler angles can be calculated by the comparison of the matrix coeficients r_{33}, r_{32}, r_{23}.

Rotation Matrix Resulting from Euler Angles

Euler angles according definition used here (Asada, Slotine):

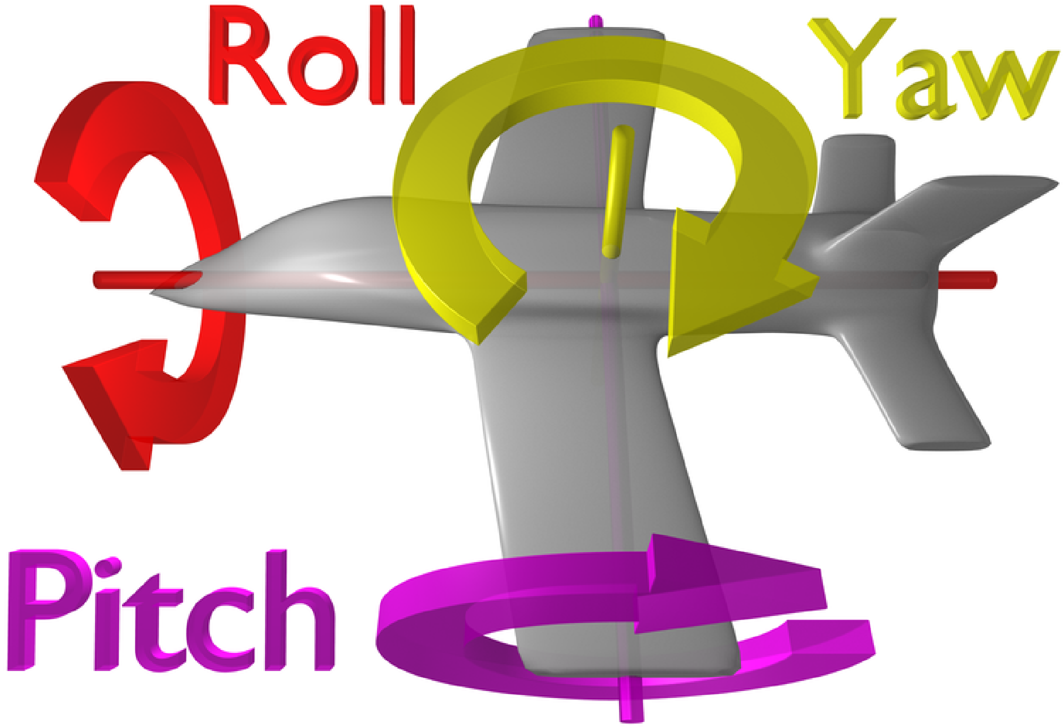
$$\begin{pmatrix} \cos\varphi\cos\psi - \cos\vartheta\sin\varphi\sin\psi & -\cos\vartheta\cos\psi\sin\varphi - \cos\varphi\sin\psi & \sin\vartheta\sin\varphi \\ \cos\psi\sin\varphi + \cos\vartheta\cos\varphi\sin\psi & \cos\vartheta\cos\varphi\cos\psi - \sin\varphi\sin\psi & -\cos\varphi\sin\vartheta \\ \sin\vartheta\sin\psi & \cos\psi\sin\vartheta & \cos\vartheta \end{pmatrix}$$

Rotation matrix based on Yaw, Pitch, Roll used in CRS robot, that is rotation around z, then y, then x:

$$\begin{pmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \cos\gamma\sin\alpha & \cos\alpha\cos\gamma\sin\beta + \sin\alpha\sin\gamma \\ \cos\beta\sin\alpha & \cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & \cos\gamma\sin\alpha\sin\beta - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{pmatrix}$$

How to calculate Euler angles when rotation matrix is known

- Let us have the known matrix $\mathbf{R}$ (3×3) and symbolic matrix composed of three rotations defined by three angles. Our task is to find those angles. The symbolic matrix has for three consecutive rotations around perpendicular axes special form, similar to the form on the above slide. So one has to solve the equation similar (not necessary indentical) to (13) with three uknowns ϕ, θ, ψ.
- First one has to find a pivot element in the symbolic matrix which is function of just one variable (monom). This element (e.g. element in the third row and third column in the example) is in the form either $\pm\cos$ or $\pm\sin$. This can be directly compared to the corresponding element in the known matrix and angle calculated. Note that generally there are two solutions in the each interval of length 2π.
- When first angle is calculated, other angles could be calculated using `atan2` function from elements on the pivot's row and column.
- One has to consider the case when the corresponding element in the constant matrix $\mathbf{R}$ has value close to ± 1. This leads to degenerate case, where in each interval of the length 2π is only one solution. The second problem is that the rest of the elements in the pivot's column and row are 0 for this degenerate case, so one cannot calculate other unknown angles from other elements in pivot's row and column. When the angle calculated from monom is inserted to the submatrix of the symbolic matrix after removing pivot's column and row, the formulas could be simplified and one will find that the submatrix is function either of sum or difference of the still unknown angles. When solving symbolically one will get one dimensional space of solutions. When solving numerically, one can either fix one angle (e.g. to 0) and calculate the other. In certain situations, one has to find other constraints for the solution. In robotics this situation indicates singular point and one can use additional information e.g where the robot arm was before or when it should move to preserve continuity of the trajectory.
- Another aspect of the numerical solution is error caused by rounding in measurement or calculations. This can easily produce for example the element of the known matrix to be larger than 1, which shall be handled appropriately.


Roll
Yaw
Pitch

Comparison of Rotation Descriptions

System	Symbol	Equivalent	Pars	Conditions
Rotation matrix	$\mathbf{R}$		9	orthonormal
Vectors of axes	$\mathbf{n}, \mathbf{t}, \mathbf{b}$	$\mathbf{R}$	9	unit vectors, orthogonal
Euler angles	ϕ, θ, ψ	yaw, pitch, roll,...	3	
Axis, angle	$\mathbf{s}, \theta$		4	unit vector
Quaternion	$\mathbf{q}$	axis, angle	4	unit vector
Rotation vector	$\mathbf{v}$	axis, angle	3	

System	Advantages	Disadvantages	Used by
$\mathbf{R}$	good for calculations	redundant	Matlab toolbox
$\mathbf{n}, \mathbf{t}, \mathbf{b}$	human understandable	redundant	
ϕ, θ, ψ	nonredundant	complicated topology	Mitsubishi Staubli, CRS
$\mathbf{s}, \theta$	human understandable easy interpolation	redundant	
$\mathbf{q}$	easy interpolation	redundant	ABB
$\mathbf{v}$	good topology nonredundant		rotation estimation

Coordinate transformation

m p
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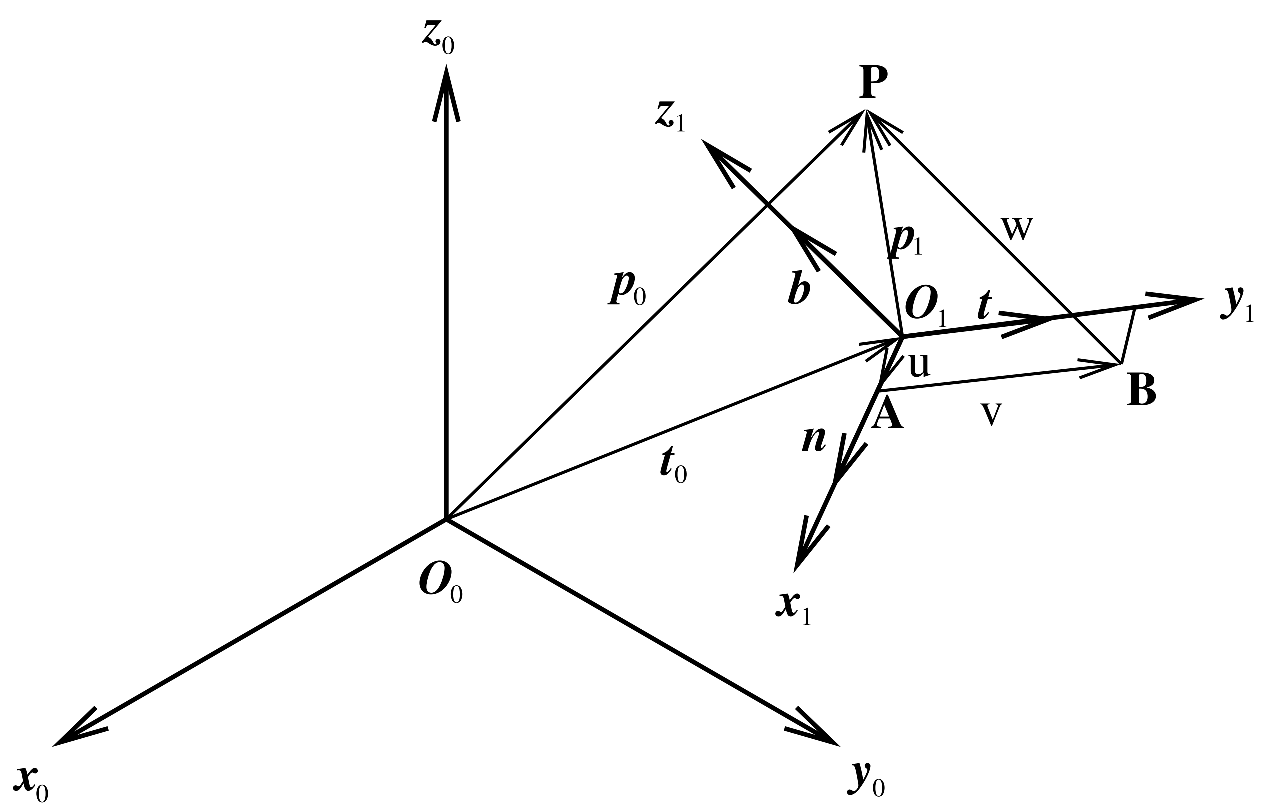


We know the coordinates of a point P in the coordinate system $O_1 - x_1y_1z_1$: $P^1 = \mathbf{p}_1^1 = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ and we are looking for its coordinates in coordinate system $O_0 - x_0y_0z_0$: $P^1 = \mathbf{p}_0^0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Lower right index refers usually to the coordinate system, to which the element relates, upper right index refers to the coordinate system in which the element is described. So origin of the coordinate system 1 (point) O_1 has coordinates in coordinate system 1: $O_1^1 = (0,0,0)^T$ but in the coordinate system 0 it has coordinates $O_1^0 = \mathbf{t}_0^0$.

Geometrically:

$$\vec{OP'} = \vec{OO'} + \vec{O'A} + \vec{AB} + \vec{BP}.$$

Algebraically:

$$P^0 = \mathbf{p}_0^0 = \mathbf{t}_0^0 + u\mathbf{n}^0 + v\mathbf{t}^0 + w\mathbf{b}^0.$$

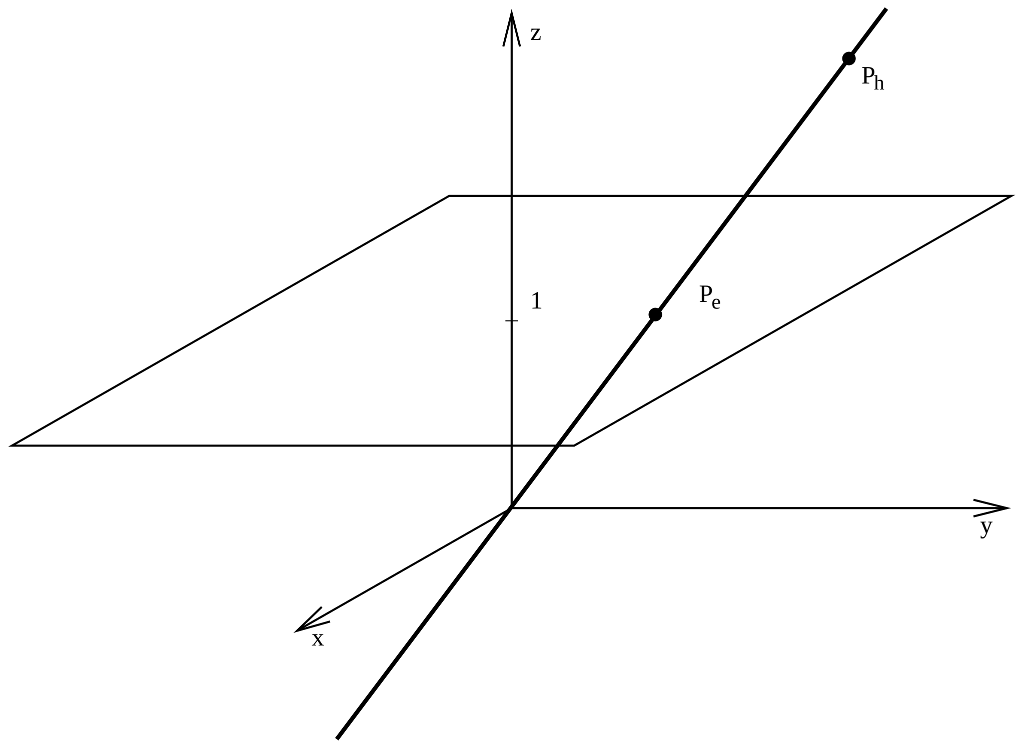
Rewritten:

$$\mathbf{p}_0^0 = \mathbf{t}_0^0 + \mathbf{R}\mathbf{p}_1^1.$$

Inverse transform:

$$\mathbf{p}_1^1 = -\mathbf{R}^T\mathbf{t}_0^0 + \mathbf{R}^T\mathbf{p}_0^0.$$

Homogeneous Coordinates



Homogeneous coordinates

Let us define homogenneous coordinates:

Euclidean -metric — Homogeneous - projective

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \Rightarrow \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x/w \\ y/w \\ z/w \end{pmatrix} \quad \Leftarrow \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \wedge w \neq 0$$

does not exist (point at infinity) $\quad \Leftarrow \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$

It is possible to show, that

$$\mathbf{x} = \mathbf{A}\mathbf{x}^b,$$

where $\mathbf{A}$ is a matrix 4x4:

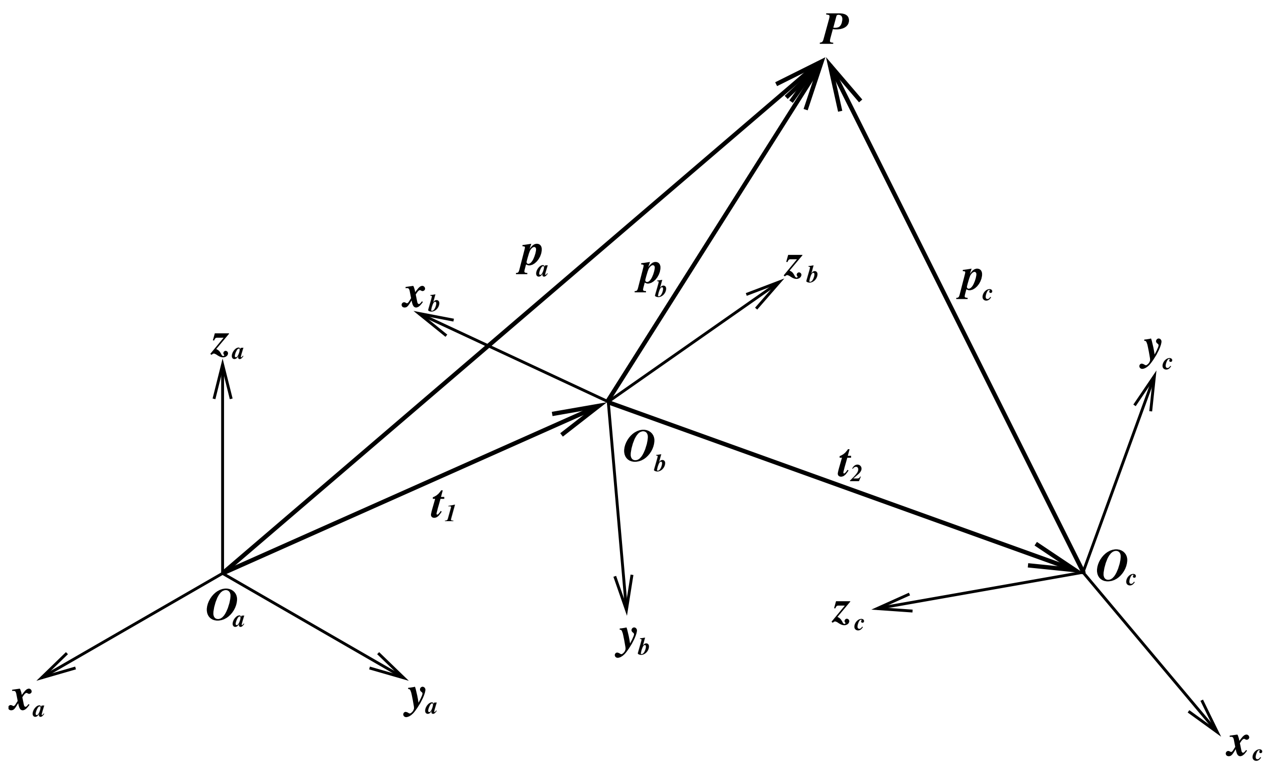
$$\mathbf{A} = \begin{bmatrix} \mathbf{R} & \mathbf{t}_0^0 \\ 0\,0\,0 & 1 \end{bmatrix}$$

Inverse matrix:

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{t}_0^0 \\ 0\,0\,0 & 1 \end{bmatrix}$$

Consecutive coordinate transformations

m p
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19



Consecutive coordinate transformations in euclidean coordinates

$$P^a = \mathbf{p}_a^a = \mathbf{t}_1^a + \mathbf{R}_b^a \mathbf{p}_b^b = \mathbf{t}_1^a + \mathbf{R}_b^a(\mathbf{t}_2^b + \mathbf{R}_c^b \mathbf{p}_c^c)$$

and in homogenneous coordinates:

$$P^a = \mathbf{p}_a^a = \mathbf{A}_b^a \mathbf{A}_c^b \mathbf{p}_c^c = \mathbf{A}_b^a \mathbf{A}_c^b P^c$$

$$P^0 = \mathbf{A}_1^0 \mathbf{A}_2^1 \mathbf{A}_3^2 \mathbf{A}_4^3 \ldots \mathbf{A}_n^{n-1} P^n. \tag{14}$$