Globally Optimal Solution to Inverse Kinematics of 7DOF Serial Manipulator

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Serial manipulator with 7DOF

▶ 7 revolute joints → 7 DOF.

▶ $i$-th joint is parametrized by angle $\theta_i$.

▶ Rigid body in space has 6DOF → redundant manipulator.

▶ One DOF left → self-motion.

Figure: Example of planar manipulator.
Direct kinematics

- Description of the manipulator by Denavit-Hartenberg (D-H) convention [2].
- D-H transformation matrices $M_i(\theta_i) \in \mathbb{R}^{4\times4}$ from link $i$ to $i-1$.
- Transformation $M$ from the end effector coordinate system to the base coordinate system

$$\prod_{i=1}^{7} M_i(\theta_i) = M. \quad (1)$$

- $M$ represents the end effector pose w.r.t. the base coordinate system

$$M = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}, \quad t \in \mathbb{R}^3 \text{ and } R \in SO(3). \quad (2)$$

- Known joint angles $\theta_i \rightarrow$ evaluation of Equation (1) gives the end effector pose $M$.
- Joint limits ($i = 1, \ldots, 7$):

$$\theta_i^{Low} \leq \theta_i \leq \theta_i^{High}. \quad (3)$$
Problem formulation

Inverse kinematics (IK) problem

- Known end effector pose $M \rightarrow$ joint angles $\theta_i$.
- Solve $\prod_{i=1}^{7} M_i(\theta_i) = M$ for $\theta_i$.
- For redundant manipulator there is an infinite number of solutions.
- Let us introduce an objective function to choose an optimal solution.

$$\min_{\theta \in (-\pi; \pi)^7} \max_{i=1}^7 \|\theta_i\|$$  \hspace{1cm} (4)

- Approximation by sum of squares.

$$\min_{\theta \in (-\pi; \pi)^7} \sum_{i=1}^{7} \theta_i^2$$  \hspace{1cm} (5)

**Figure:** Two configurations of a planar manipulator with different values of the objective function.
Optimization problem

- Optimization problem:

\[
\begin{align*}
\min_{\theta \in (-\pi; \pi)^7} & \quad \sum_{i=1}^{7} \theta_i^2 \\
\text{s.t.} & \quad \prod_{i=1}^{7} M_i(\theta_i) = M \\
& \quad \theta_i^L \leq \theta_i \leq \theta_i^H \quad (i = 1, \ldots, 7)
\end{align*}
\]

- Not polynomial, contains trigonometric functions.
- We remove them by rewriting the problem in new variables \(c = [c_1, \ldots, c_7]^\top\) and \(s = [s_1, \ldots, s_7]^\top\), which represent the cosines and sines of the joint angles \(\theta = [\theta_1, \ldots, \theta_7]^\top\) respectively.
- To preserve the structure, we need to add the trigonometric identities:

\[
q_i(c, s) = c_i^2 + s_i^2 - 1 = 0, \quad i = 1, \ldots, 7.
\]
Polynomial optimization problem

- Polynomial optimization problem equivalent to the original optimization problem:

\[
\begin{align*}
\min_{c \in \langle -1,1 \rangle^7, \; s \in \langle -1,1 \rangle^7} & \quad ||c - 1||^2 \\
s.t. & \quad p_j(c, s) = 0 \quad (j = 1, \ldots, 12) \\
& \quad q_i(c, s) = 0 \quad (i = 1, \ldots, 7) \\
& \quad -(c_i + 1) \tan \frac{\theta_i^{Low}}{2} + s_i \geq 0 \quad (i = 1, \ldots, 7) \\
& \quad (c_i + 1) \tan \frac{\theta_i^{High}}{2} - s_i \geq 0 \quad (i = 1, \ldots, 7)
\end{align*}
\]

- In 14 variables \((c\) and \(s\)).
- Contains polynomials up to degree four.
- When solved, \(\theta\) are recovered from \(c\) and \(s\) by function \text{atan2}.
Direct application of polynomial solver


- Second order relaxation
  - 14 variables, degree 4 polynomials $\rightarrow$ SDP program with 3060 variables.
  - Computation time in seconds.
  - Solution not obtained in many cases.

- Third order relaxation
  - 14 variables, degree 6 polynomials $\rightarrow$ SDP program with 38760 variables.
  - Computation time in hours.
Symbolic reduction

Theorem

The ideal generated by the kinematics constraints $p_j$ for generic serial manipulator with seven revolute joints and for generic pose $M$ with addition of the trigonometric identities $q_i$ can be generated by a set of degree two polynomials.

Proof.

The proof is computational. See the diagram.

\[ G \leftarrow \text{Gröbner basis of } \langle p_j, q_i \rangle \]

\[ S = \{ f \in G \mid \deg(f) = 2 \} \]

\[ G' \leftarrow \text{Gröbner basis of } S \]

\[ G = G' \]

\[ \langle p_j, q_i \rangle = \langle S \rangle \]
Solving the reduced polynomial optimization problem

Corollary

Polynomials \( p_j \) and \( q_i \) up to degree four in POP can be replaced by degree two polynomials.

- First order relaxation
  - 14 variables, degree 2 polynomials → SDP program with 120 variables.
  - Solution typically not obtained.
- Second order relaxation
  - 14 variables, degree 4 polynomials → SDP program with 3060 variables.
  - Computation time in seconds.
  - Gives solution for almost all poses.
Experiments with KUKA LBR iiwa

- Special structure: for fixed end effector pose the joint angle $\theta_4$ is constant.

- Previous work:
  - Geometrical derivation of a closed form solution by Kuhlemann et al. [3]; new parameter $\delta$ is introduced to fix the left DOF.
  - Dai et al. [1] proposed mix-integer convex relaxation of the non-convex rotational constraints; approximation introduces errors in units of centimeters and degrees.

- Synthetic dataset:
  - 10,000 randomly chosen poses.
  - From within and outside of the working space of the manipulator.
Experiments

Degree four polynomials

- Solve the polynomial optimization problem with degree four polynomials.
- For relaxation order two.
- Using polynomial optimization toolbox GloptiPoly with MOSEK as the semidefinite problem solver.
- For 29.3% poses we failed to compute the solution or report infeasibility.

Figure: Poses of the manipulator solved from degree four polynomials.
Degree two polynomials

- Advantage of special structure of KUKA LBR: eliminate variables $c_4$ and $s_4$.

- Symbolically reduce the degree four polynomials to degree two polynomials (Maple).

- Solve for relaxation order two.

- Using polynomial optimization toolbox GloptiPoly with MOSEK as the semidefinite problem solver.

- For 0.1% poses we failed to compute the solution or report infeasibility.

Figure: Poses of the manipulator solved from degree two polynomials.
Experiments

Numerical stability and execution time evaluation

- End effector poses have been computed by direct kinematics from estimated $\theta$.
- Pose error w.r.t. desired poses measured in 3D space.
- Execution time of on-line phase of GloptiPoly and of the symbolic reduction of the polynomials.

Figure: Histogram of pose errors.

Figure: Histograms of execution time.
Conclusions

- We proved that the variety of IK solutions of all generic 7DOF revolute serial manipulators can be generated by second degree polynomials only.
- We presented a practical method for globally solving 7DOF IK problem with polynomial objective function.
- Our solution is accurate and can solve/decide infeasibility in 99.9% cases tested on KUKA LBR iiwa manipulator.
- The code is open-sourced at https://github.com/PavelTrutman/Global-7DOF-IKT.

<table>
<thead>
<tr>
<th>Execution time [s]</th>
<th>Median error [mm]</th>
<th>Median error [deg]</th>
<th>% of failed poses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reduction step</td>
<td>Translation</td>
<td>Rotation</td>
<td></td>
</tr>
<tr>
<td>—</td>
<td>Deg. 4 polynomials</td>
<td>2.12 $\cdot 10^{-4}$</td>
<td>3.32 $\cdot 10^{-5}$</td>
</tr>
<tr>
<td>2.3</td>
<td>Deg. 2 polynomials</td>
<td>6.28 $\cdot 10^{-5}$</td>
<td>5.57 $\cdot 10^{-3}$</td>
</tr>
<tr>
<td>5.5</td>
<td></td>
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</table>

Table: Overview of execution times and accuracy of the presented methods.
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Bibliography

Global inverse kinematics via mixed-integer convex optimization.

A kinematic notation for lower pair mechanisms based on matrices.

Robust inverse kinematics by configuration control for redundant manipulators with seven dof.

Global optimization with polynomials and the problem of moments.