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Globally Optimal Solution to Inverse Kinematics of 7DOF Serial Manipulator

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Globally Optimal Solution to Inverse Kinematics of 7DOF Serial Manipulator

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Serial manipulator with 7 DOF

- 7 revolute joints → 7 DOF.
- $i$-th joint is parametrized by angle $\theta_i$.
- Rigid body in space has 6 DOF → redundant manipulator.
- One DOF left → self-motion.

Figure: Example of planar manipulator.
Denavit-Hartenberg convention

- Description of the manipulator by Denavit-Hartenberg (D-H) convention [HD55].

- Parameters $\alpha_i$, $d_i$ and $a_i$ are found (fixed for given manipulator).

- D-H transformation matrices $M_i(\theta_i) \in \mathbb{R}^{4 \times 4}$ from link $i$ to $i - 1$.

$$M_i(\theta_i) = \begin{bmatrix} \cos \theta_i & -\sin \theta_i & 0 & 0 \\ \sin \theta_i & \cos \theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos \alpha_i & -\sin \alpha_i & 0 \\ 0 & \sin \alpha_i & \cos \alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ (1)
Forward kinematics

- Transformation $M$ from the end effector coordinate system to the base coordinate system

$$\prod_{i=1}^{7} M_i(\theta_i) = M. \tag{2}$$

- $M$ represents the end effector pose w.r.t. the base coordinate system

$$M = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}, \quad t \in \mathbb{R}^3 \text{ and } R \in SO(3). \tag{3}$$

- Known joint angles $\theta_i \rightarrow$ evaluation of Equation (2) gives the end effector pose $M$.

- Joint limits ($i = 1, \ldots, 7$):

$$\theta_i^{Low} \leq \theta_i \leq \theta_i^{High}. \tag{4}$$
Inverse kinematics (IK) problem

- Known end effector pose \( M \rightarrow \text{joint angles} \ \theta_i \).
- Solve \( \prod_{i=1}^{7} M_i(\theta_i) = M \) for \( \theta_i \).
- For redundant manipulator there is an infinite number of solution.
- Let us introduce an objective function to choose an optimal solution.

\[
\min_{\theta \in (-\pi; \pi)^7} \max_{i=1}^{7} \|\theta_i\| \tag{5}
\]

- Approximation by sum of squares.

\[
\min_{\theta \in (-\pi; \pi)^7} \sum_{i=1}^{7} \theta_i^2 \tag{6}
\]

**Figure:** Two configurations of a planar manipulator with different values of the objective function.
Optimization problem

- Optimization problem:

\[
\begin{align*}
\min_{\theta \in (-\pi; \pi)^7} & \sum_{i=1}^{7} \theta_i^2 \\
\text{s.t.} & \prod_{i=1}^{7} M_i(\theta_i) = M \\
& \theta_{i\text{Low}} \leq \theta_i \leq \theta_{i\text{High}} \quad (i = 1, \ldots, 7)
\end{align*}
\]  

- Not polynomial, contains trigonometric functions.

- We remove them by rewriting the problem in new variables \( c = [c_1, \ldots, c_7]^\top \) and \( s = [s_1, \ldots, s_7]^\top \), which represent the cosines and sines of the joint angles \( \theta = [\theta_1, \ldots, \theta_7]^\top \) respectively.

- To preserve the structure, we need to add the trigonometric identities:

\[
q_i(c, s) = c_i^2 + s_i^2 - 1 = 0, \quad i = 1, \ldots, 7.
\]
Problem formulation

Polynomial optimization problem

▶ Polynomial optimization problem equivalent to the original optimization problem:

\[
\begin{align*}
\min_{c \in \langle -1, 1 \rangle^7, \ s \in \langle -1, 1 \rangle^7} & \quad ||c - 1||^2 \\
\text{s.t.} & \quad p_j(c, s) = 0 \quad (j = 1, \ldots, 12) \\
& \quad q_i(c, s) = 0 \quad (i = 1, \ldots, 7) \\
& \quad -(c_i + 1) \tan \frac{\theta_i^{Low}}{2} + s_i \geq 0 \quad (i = 1, \ldots, 7) \\
& \quad (c_i + 1) \tan \frac{\theta_i^{High}}{2} - s_i \geq 0 \quad (i = 1, \ldots, 7)
\end{align*}
\]  

(9)

▶ In 14 variables \((c\) and \(s\)).
▶ Contains polynomials up to degree four.
▶ When solved, \(\theta\) are recovered from \(c\) and \(s\) by function \(\text{atan2}\).
Polynomial optimization methods

Polynomial problem:
- Objective function: polynomial.
- Constraints: polynomial inequalities and equations.
- Non-convex.

Semidefinite program [Las01]:
- Each monomial is substituted by a new variable.
- Objective function: linear.
- Constraints: linear matrix inequalities, linear equations.
- Convex, but infinite-dimensional.
Polynomial optimization methods

Polynomial problem:
- Objective function: polynomial.
- Constraints: polynomial inequalities and equations.
- Non-convex.

Relaxed semidefinite program [Las01]:
- Limit the degree of substituted monomials by degree $r \in \mathbb{N}$.
- Convex and finite-dimensional.
- Convergence is ensured.

Implemented in Gloptipoly [HLL09].

\[ p_r^* \leq p_{r+1}^* \leq p^* \quad (10) \]
\[ \lim_{r \to +\infty} p_r^* = p^* \quad (11) \]
Direct application of polynomial solver

- Direct application of Lasserre hierarchies [Las01] on the problem.

- Second order relaxation
  - 14 variables, monomials up to degree 4 → SDP program with 3060 variables.
  - Computation time in seconds.
  - Solution not obtained in many cases.

- Third order relaxation
  - 14 variables, monomials up to degree 6 → SDP program with 38,760 variables.
  - Computation time in hours.
Symbolic reduction

Theorem

The ideal generated by the kinematics constraints $p_j$ for generic serial manipulator with seven revolute joints and for generic pose $M$ with addition of the trigonometric identities $q_i$ can be generated by a set of degree two polynomials.

Proof.

The proof is computational. See the diagram.
Solving the reduced polynomial optimization problem

Corollary

Polynomials $p_j$ and $q_i$ up to degree four in POP can be replaced by degree two polynomials.

- Application of Lasserre hierarchies [Las01] on the symbolically reduced problem with degree two polynomials.

- First order relaxation
  - 14 variables, monomials up to degree 2 → SDP program with 120 variables.
  - Solution typically not obtained.

- Second order relaxation
  - 14 variables, monomials up to degree 4 → SDP program with 3060 variables.
  - Computation time in seconds.
  - Gives solution for almost all poses.
Experiments with KUKA LBR iiwa

- Special structure: for fixed end effector pose the joint angle $\theta_4$ is constant.

- Previous work:
  - Geometrical derivation of a closed form solution by Kuhlemann et al. [Kuh+16]; new parameter $\delta$ is introduced to fix the left DOF.
  - Dai et al. [DIT17] proposed mix-integer convex relaxation of the non-convex rotational constraints; approximation introduces errors in units of centimeters and degrees.

- Synthetic dataset:
  - 10 000 randomly chosen poses.
  - From within and outside of the working space of the manipulator.

Figure: Manipulator KUKA LBR iiwa.
Experiments

Degree four polynomials

- Solve the polynomial optimization problem with degree four polynomials.
- For relaxation order two.
- Using polynomial optimization toolbox GloptiPoly with MOSEK as the semidefinite problem solver.
- For 29.3% poses we failed to compute the solution or report infeasibility.

Figure: Poses of the manipulator solved from degree four polynomials.
Degree two polynomials

- Advantage of special structure of KUKA LBR: eliminate variables $c_4$ and $s_4$.

- Symbolically reduce the degree four polynomials to degree two polynomials (Maple).

- Solve for relaxation order two.

- Using polynomial optimization toolbox GloptiPoly with MOSEK as the semidefinite problem solver.

- For 0.1 % poses we failed to compute the solution or report infeasibility.

Figure: Poses of the manipulator solved from degree two polynomials.
Experiments

Numerical stability and execution time evaluation

- End effector poses have been computed by direct kinematics from estimated $\theta$.
- Pose error w.r.t. desired poses measured in 3D space.

Execution time of on-line phase of GloptiPoly and of the symbolic reduction of the polynomials.

![Histogram of pose errors.](image1)

![Histograms of execution time.](image2)
Conclusions

- We proved that the variety of IK solutions of all generic 7DOF revolute serial manipulators can be generated by second degree polynomials only.
- We presented a practical method for globally solving 7DOF IK problem with polynomial objective function.
- Our solution is accurate and can solve/decide infeasibility in 99.9 % cases tested on KUKA LBR iiwa manipulator.
- The code is open-sourced at https://github.com/PavelTrutman/Global-7DOF-IKT.

<table>
<thead>
<tr>
<th>Execution time [s]</th>
<th>Median error</th>
<th>% of failed poses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reduction step</td>
<td>Translation [mm]</td>
<td>Rotation [deg]</td>
</tr>
<tr>
<td>Deg. 4 polynomials</td>
<td>—</td>
<td>2.12 \times 10^{-4}</td>
</tr>
<tr>
<td>Deg. 2 polynomials</td>
<td>2.3</td>
<td>6.28 \times 10^{-5}</td>
</tr>
</tbody>
</table>

Table: Overview of execution times and accuracy of the presented methods.
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Bibliography I


