Globally Optimal Solution to Inverse Kinematics of 7DOF Serial Manipulator

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October 21, 2020
Serial manipulator with 7 DOF

- 7 revolute joints $\rightarrow$ 7 DOF.

- $i$-th joint is parameterized by angle $\theta_i$.

- Rigid body in space has 6 DOF $\rightarrow$ redundant manipulator.

- One DOF left $\rightarrow$ self-motion.

Figure: Example of planar manipulator.
Forward kinematics

- Description of the manipulator by Denavit-Hartenberg (D-H) convention [HD55].
- D-H transformation matrices $M_i(\theta_i) \in \mathbb{R}^{4 \times 4}$ from link $i$ to $i-1$.
- Transformation $M$ from the end effector coordinate system to the base coordinate system

$$\prod_{i=1}^{7} M_i(\theta_i) = M. \quad (1)$$

- $M$ represents the end effector pose w.r.t. the base coordinate system

$$M = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}, \quad t \in \mathbb{R}^3 \text{ and } R \in SO(3). \quad (2)$$

- Known joint angles $\theta_i \rightarrow$ evaluation of Equation (1) gives the end effector pose $M$.
- Joint limits ($i = 1, \ldots, 7$):

$$\theta_i^{Low} \leq \theta_i \leq \theta_i^{High}. \quad (3)$$
Inverse kinematics (IK) problem

- Known end effector pose $M \rightarrow$ joint angles $\theta_i$.
- Solve $\prod_{i=1}^{7} M_i(\theta_i) = M$ for $\theta_i$.
- For redundant manipulator there is an infinite number of solution.
- Let us introduce an objective function to choose an optimal solution:

$$\min_{\theta \in (-\pi; \pi)^7} \sum_{i=1}^{7} w_i \left( (\theta_i - \hat{\theta}_i) \mod \pi \right) \quad (4)$$

with weights $w_i > 0$, $\sum_{i=1}^{7} w_i = 1$ and where $\hat{\theta}$ are preferred values for $\theta$.

Figure: Two configurations of a planar manipulator with different values of the objective function for given $\hat{\theta}$.
**Optimization problem**

- Optimization problem:

  \[
  \min_{\theta \in (-\pi; \pi)^7} \sum_{i=1}^{7} w_i \left( (\theta_i - \hat{\theta}_i) \mod \pi \right)
  \]

  s.t. \( \prod_{i=1}^{7} M_i(\theta_i) = M \)
  \( \theta_i^{Low} \leq \theta_i \leq \theta_i^{High} \) (\( i = 1, \ldots, 7 \))

- Not polynomial, contains trigonometric functions.

- We remove them by rewriting the problem in new variables \( c = [c_1, \ldots, c_7]^T \) and \( s = [s_1, \ldots, s_7]^T \), which represent the cosines and sines of the joint angles \( \theta = [\theta_1, \ldots, \theta_7]^T \) respectively.

- To preserve the structure, we need to add the trigonometric identities:

  \[
  q_i(c, s) = c_i^2 + s_i^2 - 1 = 0, \ i = 1, \ldots, 7.
  \]
Problem formulation

Polynomial optimization problem

- Polynomial optimization problem equivalent to the original optimization problem:

\[
\min_{c \in \langle -1,1 \rangle^7, \ s \in \langle -1,1 \rangle^7} \sum_{i=1}^{7} 2w_i (1 - c_i \cos \hat{\theta}_i - s_i \sin \hat{\theta}_i)
\]

s.t.

\[
p_j(c, s) = 0 \quad (j = 1, \ldots, 12)
\]

\[
q_i(c, s) = 0 \quad (i = 1, \ldots, 7)
\]

\[-(c_i + 1) \tan \frac{\theta_{i, \text{Low}}}{2} + s_i \geq 0 \quad (i = 1, \ldots, 7)
\]

\[(c_i + 1) \tan \frac{\theta_{i, \text{High}}}{2} - s_i \geq 0 \quad (i = 1, \ldots, 7)
\]

- In 14 variables \((c\) and \(s\)).
- Contains polynomials up to degree four.
- When solved, \(\theta\) are recovered from \(c\) and \(s\) by function \(\text{atan2}\).
Polynomial optimization methods

Polynomial problem:
- Objective function: polynomial.
- Constraints: polynomial inequalities and equations.
- Non-convex.

Semidefinite program [Las01]:
- Each monomial is substituted by a new variable.
- Objective function: linear.
- Constraints: linear matrix inequalities, linear equations.
- Convex, but infinite-dimensional.
Polynomial optimization methods

Polynomial problem:

- Objective function: polynomial.
- Constraints: polynomial inequalities and equations.
- Non-convex.

Relaxed semidefinite program [Las01]:

- Limit the degree of substituted monomials by degree $r \in \mathbb{N}$.
- Convex and finite-dimensional.
- Convergence is ensured.

Implemented in Gloptipoly [HLL09].

\begin{align*}
    p_r^* & \leq p_{r+1}^* \leq p^* \\
    \lim_{r \to +\infty} p_r^* & = p^* 
\end{align*} 

\( p_r^* \leq p_{r+1}^* \leq p^* \)  \( \lim_{r \to +\infty} p_r^* = p^* \)
Direct application of polynomial solver

- Direct application of Lasserre hierarchies [Las01] on the problem.

- Second order relaxation
  - 14 variables, monomials up to degree 4 → SDP program with 3060 variables.
  - Computation time in seconds.
  - Solution not obtained in many cases.

- Third order relaxation
  - 14 variables, monomials up to degree 6 → SDP program with 38760 variables.
  - Computation time in hours.
Symbolic reduction

Theorem

The ideal generated by the kinematics constraints $p_j$ for generic serial manipulator with seven revolute joints and for generic pose $M$ with addition of the trigonometric identities $q_i$ can be generated by a set of degree two polynomials.

Proof.

The proof is computational. See the diagram.  

\[ G = G' \]

\[ \langle p_j, q_i \rangle = \langle S \rangle \]
Solving the reduced polynomial optimization problem

Corollary

*Polynomials* $p_j$ and $q_i$ *up to degree four in POP can be replaced by degree two polynomials.*

- Application of Lasserre hierarchies [Las01] on the symbolically reduced problem with degree two polynomials.

- **First order relaxation**
  - 14 variables, monomials up to degree 2 $\rightarrow$ SDP program with 120 variables.
  - Solution typically not obtained.

- **Second order relaxation**
  - 14 variables, monomials up to degree 4 $\rightarrow$ SDP program with 3060 variables.
  - Computation time in seconds.
  - Gives solution for almost all poses.
Experiments with KUKA LBR iiwa

- Special structure: for fixed end effector pose the joint angle $\theta_4$ is constant.

- Previous work:
  - Geometrical derivation of a closed form solution by Kuhlemann et al. [Kuh+16]; new parameter $\delta$ is introduced to fix the left DOF.
  - Dai et al. [DIT17] proposed mix-integer convex relaxation of the non-convex rotational constraints; approximation introduces errors in units of centimeters and degrees.

- Synthetic dataset:
  - 10,000 randomly chosen poses.
  - From within and outside of the working space of the manipulator.

- Solved for $\hat{\theta}_i = 0$ and $w_i = w_j$ for $i, j = 1, \ldots, 7$. 

Figure: Manipulator KUKA LBR iiwa.
Degree four polynomials

- Solve the polynomial optimization problem with degree four polynomials.
- For relaxation order two.
- Using polynomial optimization toolbox GloptiPoly with MOSEK as the semidefinite problem solver.
- For 32.4% poses we failed to compute the solution or report infeasibility.

Figure: Poses of the manipulator solved from degree four polynomials.
Degree two polynomials

- Advantage of special structure of KUKA LBR: eliminate variables $c_4$ and $s_4$.

- Symbolically reduce the degree four polynomials to degree two polynomials (Maple).

- Solve for relaxation order two.

- Using polynomial optimization toolbox GloptiPoly with MOSEK as the semidefinite problem solver.

- For 1.2% poses we failed to compute the solution or report infeasibility.

Figure: Poses of the manipulator solved from degree two polynomials.
Experiments

Numerical stability and execution time evaluation

- End effector poses have been computed by direct kinematics from estimated $\theta$.
- Pose error w.r.t. desired poses measured in 3D space.

![Histogram of pose errors.](image)

**Figure:** Histogram of pose errors.

- Execution time of on-line phase of GloptiPoly and of the symbolic reduction of the polynomials.

![Histograms of execution time.](image)

**Figure:** Histograms of execution time.
Conclusions

▶ We proved that the variety of IK solutions of all generic 7DOF revolute serial manipulators can be generated by second degree polynomials only.
▶ We presented a practical method for globally solving 7DOF IK problem with polynomial objective function.
▶ Our solution is accurate and can solve/decide infeasibility in 99 % cases tested on KUKA LBR iiwa manipulator.
▶ The code is open-sourced at https://github.com/PavelTrutman/Global-7DOF-IKT.

<table>
<thead>
<tr>
<th>Execution time [s]</th>
<th>Median error</th>
<th>% of failed poses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reduction step</td>
<td>Translation [mm]</td>
<td>Rotation [deg]</td>
</tr>
<tr>
<td>GloptiPoly</td>
<td>Deg. 4 polynomials</td>
<td>Deg. 2 polynomials</td>
</tr>
<tr>
<td>—</td>
<td>21.3</td>
<td>2.7</td>
</tr>
<tr>
<td>3.92 \cdot 10^{-4}</td>
<td>6.12 \cdot 10^{-5}</td>
<td>32.4 %</td>
</tr>
<tr>
<td>6.59 \cdot 10^{-3}</td>
<td>1.2 %</td>
<td></td>
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</tbody>
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Table: Overview of execution times and accuracy of the presented methods.
Acknowledgement

P. Trutman was supported by the EU Structural and Investment Funds, Operational Programe Research, Development and Education under the project IMPACT (reg. no. CZ.02.1.01/0.0/0.0/15_003/0000468) and Grant Agency of the CTU Prague project SGS19/173/OHK3/3T/13.

T. Pajdla was supported by IMPACT Project CZ.02.1.01/0.0/0.0/15_003/0000468 & EU Structural and Investment Funds, Operational Programe Research, Development and Education and ARtwin - An AR cloud and digital twins solution for industry and construction 4.0 (GA No 856994) H-2020 project.

