Optimization methods, an overview

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Outline of the talk:

- Optimization, task formulation.
- Modeling/optimization interplay.
- Two steps: (a) model; (b) problem type.

(Mathematical) Optimization

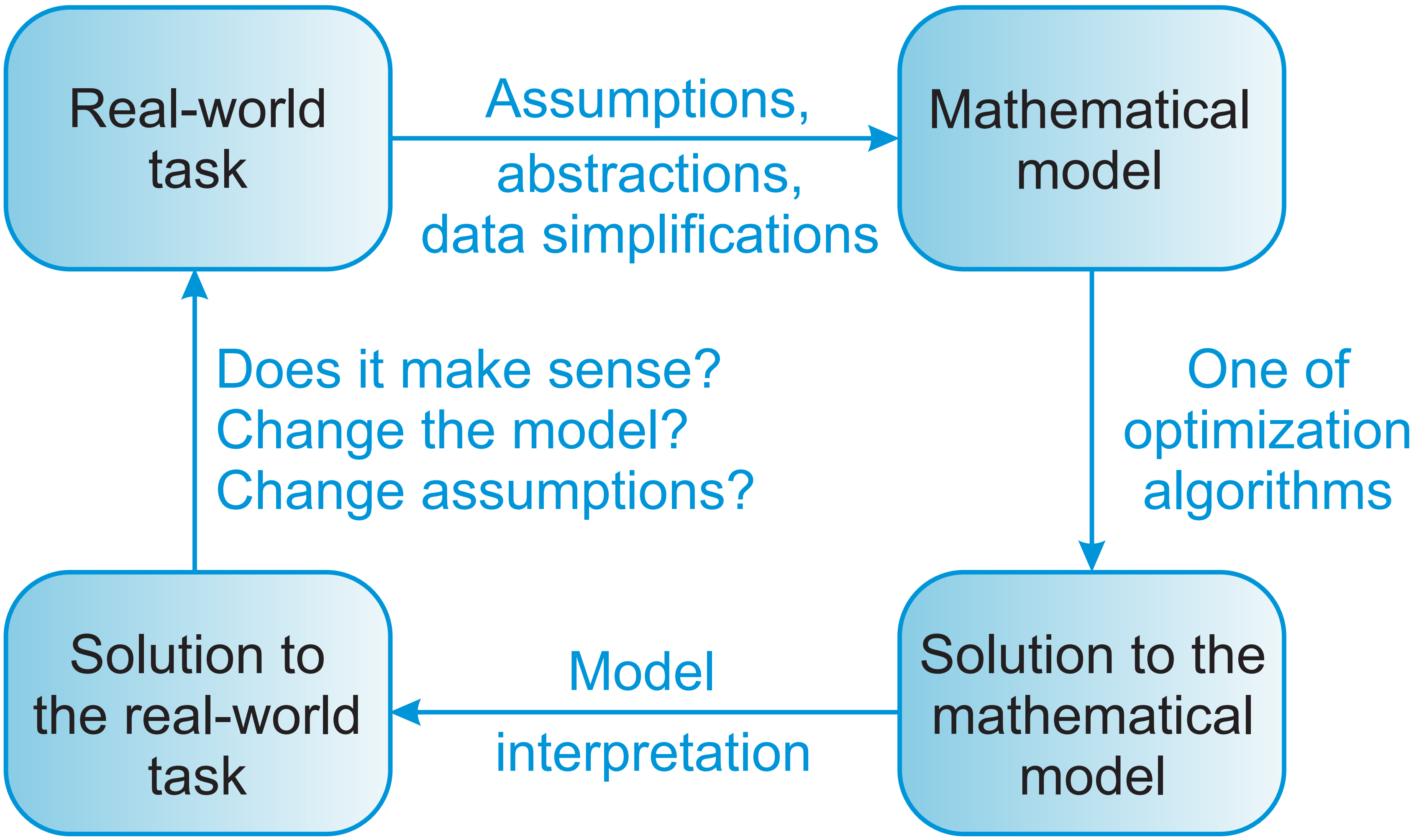
Optimization (Oxford dictionary): The action of making the best or most effective use of a situation or resource.

Mathematical optimization or mathematical programming *(in mathematics, computer science and operations research)* is

- the selection of a best element from some set of available alternatives
- with regard to some criterion.

Engineering optimization uses optimization techniques to aid the engineering design process.

A schematic view of a modeling/optimization process



1st step in optimization = constructing a model

Modeling is the process of identifying and expressing in mathematical terms the objective, the variables, and the constraints of the problem.

Objective is a quantitative measure of the performance of the system that we want to minimize or maximize.

In manufacturing, we may want to maximize the profits or minimize the cost of production. In fitting experimental data to a model, we may want to minimize the total deviation of the observed data from the predicted data.

Variables or the unknowns are the components of the system for which we want to find values.

In manufacturing, the variables may be the amount of each resource consumed or the time spent on each activity. In data fitting, the variables would be the parameters of the model.

Constraints are the functions that describe the relationships among the variables and that define the allowable values for the variables.

Optimization problem, a general mathematical formulation

- Given:

  a function $f: A \to \mathbb{R}$, i.e. from some set A to the real numbers $\mathbb{R}$.

  Function f is called, variously, objective function, utility function, cost function, loss function, penalty, fitness function, energy function, ...

- Sought:

  an element $x_0 \in A$ such that

  - $f(x_0) \leq f(x)$ for all $x \in A$ ('minimization') or
  - $f(x_0) \geq f(x)$ for all $x \in A$ ('maximization').

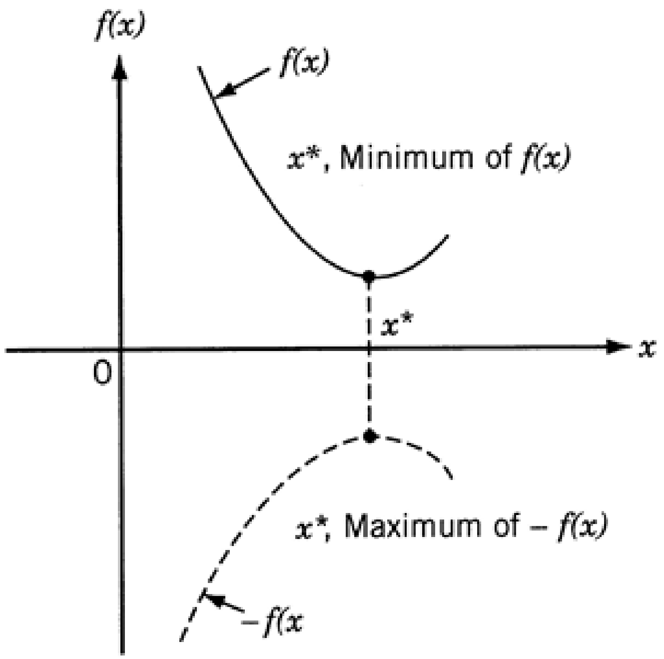
Usually, set A is some subset or the Euclidean space $\mathbb{R}^n$, often specified by a set of constraints, equalities or inequalities.

The domain A of f is called the search space or choice set.

Elements of A are called feasible solutions or candidate solutions.

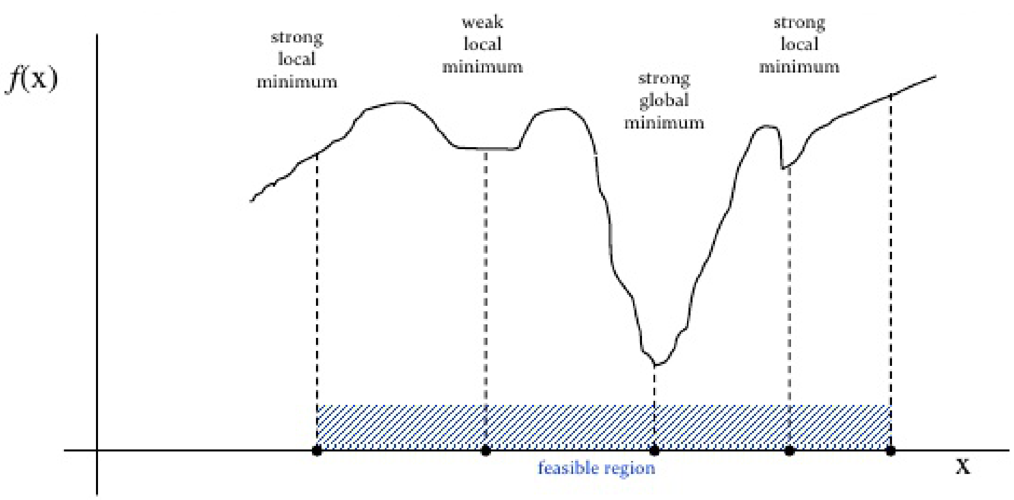
Optimization problem, mathematically, cont.

- For simplification, the minimization is often considered. If a point x^* minimizes the objective function $f(x)$, the same point also maximizes , $-f(x)$.



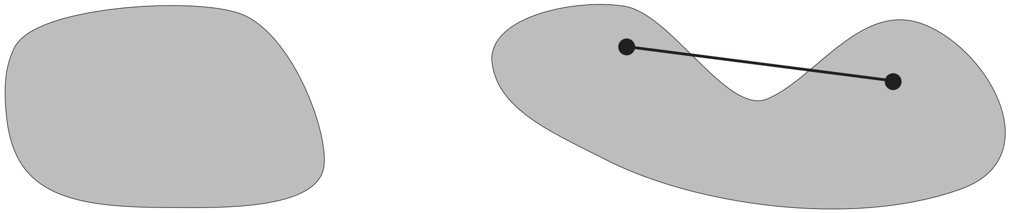
- Local minima: Generally, unless both the objective function and the feasible region are convex in a minimization problem, there may be several local minima.

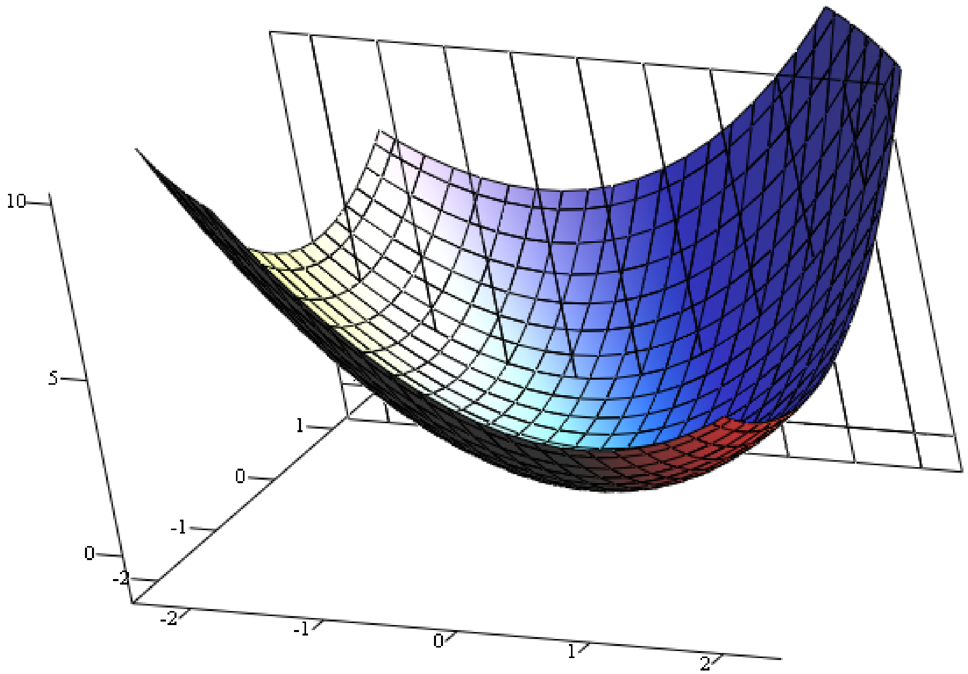
- A local minimum x^* is defined as a point for which there exists some $\delta > 0$ such that for all x where $\|x - x^*\| \leq \delta$ the expression $f(x^*) \leq f(x)$ holds.



Convex optimization task

- Convex minimization (optimization) is a subfield of optimization that studies the problem of minimizing convex functions over convex sets.
- The convexity makes optimization easier than the general case since local minimum must be a global minimum, and first-order conditions are sufficient conditions for optimality.
- Set $S \subseteq \mathbb{R}^n$ is convex if it contain a line segment between any two its points.
- Function $f: S \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex on a convex set if its graph along any line segment in S lies on or below the chord connecting function values at endpoints of the segment.





2nd step in optimization = determining the problem type

Note: The community have not come to a clear taxonomy of optimization methods yet.

- Continuous vs. discrete optimization.

  The division is based on the character of variables. Continuous methods are easier than discrete methods.

- Static (=optimizes parameters) vs. dynamic (=optimizes shape of the function) optimization.

- Unconstrained vs. constrained optimization.

  The division tackles variables. E.g. simple bounds; system of equalities and inequalities; more complex relations. The nature of constraints: e.g. linear, nonlinear, convex; differentiable vs. nondifferentiable.

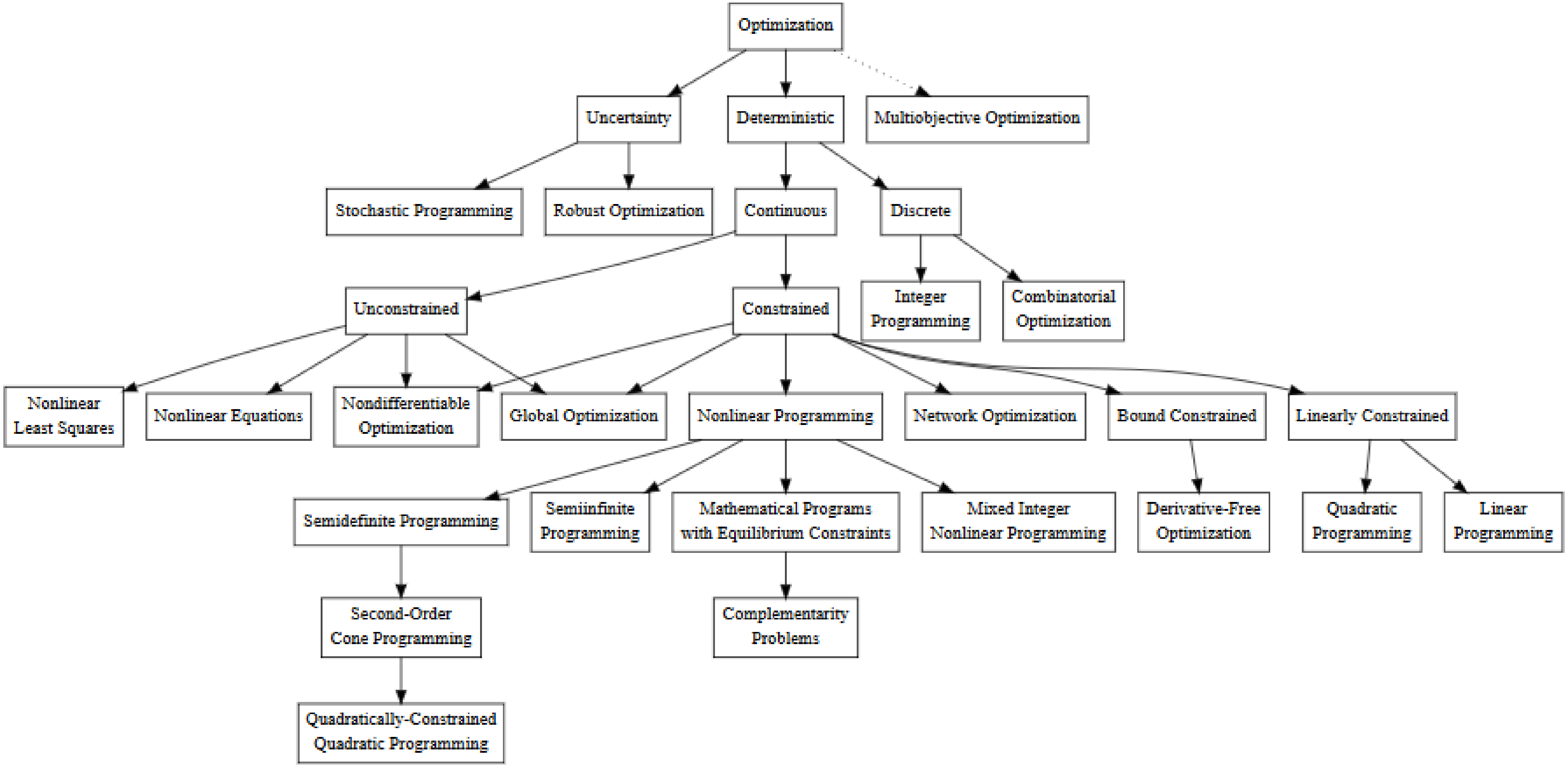
- Deterministic optimization vs. optimization under uncertainty.

  Deterministic methods assume crisp situations. Often data cannot be known accurately. Reason 1: A simple measurement error. Reason 2: Some data represent information about the future (e. g., future temperature) and simply cannot be known with certainty. In optimization under uncertainty, or stochastic optimization, the uncertainty is incorporated into the model.

- None, one objective or many objectives.

  Most optimization problems have a single objective function, however, there are interesting cases when optimization problems have no objective function or multiple objective functions.

One possible taxonomy of optimization methods



Source: https://neos-guide.org/content/optimization-taxonomy