Grayscale mathematical morphology

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Courtesy: Petr Matula, Petr Kodl, Jean Serra, Miroslav Svoboda

Outline of the talk:

- Set-function equivalence.
- Umbra and top of a set.
- Gray scale dilation, erosion.
- Top-hat transform.
- Geodesic method. Ultimate erosion.
- Morphological reconstruction.

A quick informal explanation

- Grayscale mathematical morphology is the generalization of binary morphology for images with more gray levels than two or with voxels.
- The point set $A \in \mathbb{E}^3$. The first two coordinates span in the function (point set) domain and the third coordinate corresponds to the function value.
- The concepts supremum $\vee$ (also the least upper bound), resp. infimum $\wedge$ (also the greatest lower bound) play a key role here. Actually, the related operators $\max$, resp. $\min$, are used in computations with finite sets.
- Erosion (resp. dilation) of the image (with the flat structuring) element assigns to each pixel the minimal (resp. maximal) value in the chosen neighborhood of the current pixel of the input image.
- The structuring element (function) is a function of two variables. It influences how pixels in the neighborhood of the current pixel are taken into account. The value of the (non-flat) structuring element is added (while dilating), resp. subtracted (while eroding) when the maximum, resp. minimum is calculated in the neighborhood.

Grayscale mathematical morphology explained via binary morphology

- It is possible to introduce grayscale mathematical morphology using the already explained binary (black and white only) mathematical morphology.

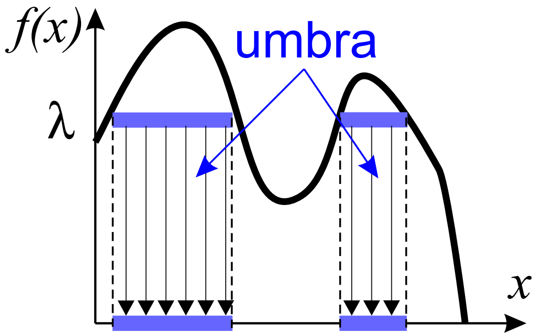
  R.M. Haralick, S.R. Sternberg, X. Zhuang: Image analysis using mathematical morphology, IEEE Pattern Analysis and Machine Intelligence, Vol. 9, No. 4, 1987, pp. 532-550.

- We will start with this explanation first and introduce an alternative way using $\sup$, $\inf$ later.
- We have to explain the concepts top of the surface and umbra first.

Equivalence between sets and functions

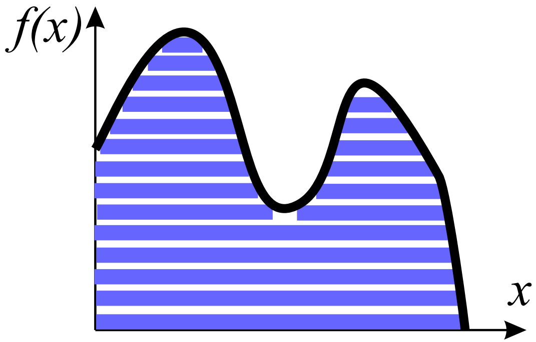
- A function can be viewed as a stack of decreasing sets. Each set X_λ is the intersection between the umbra of the function and a horizontal plane (line).

$$X_\lambda = \{X \in \mathbb{E},\ f(x) \geq \lambda\}$$
$$\Rightarrow f(x) = \sup\{\lambda : x \in X_\lambda(f)\}$$



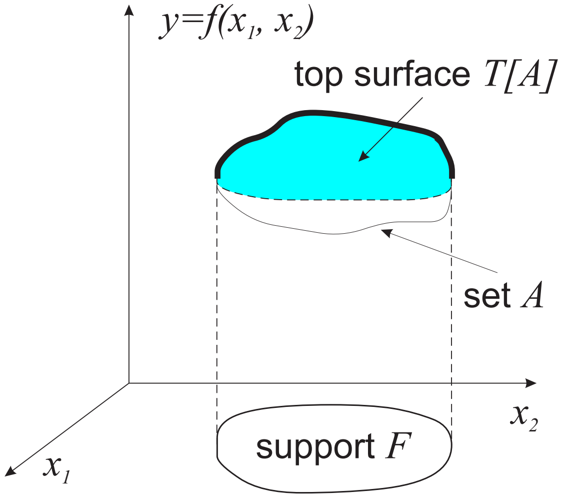
A function to sets

- It is equivalent to say that f is upper semi-continuous or that X_λ-s are closed.
- Conversely, given $\{X_\lambda\}$ of closed set such that $\lambda \geq \mu \Rightarrow X_\lambda \subseteq X_\mu$ and $X_\lambda = \cap\{X_\mu,\ \mu < \lambda\}$ then there exist a unique an upper semi-continuous. function f whose sections are $\{X_\lambda\}$.

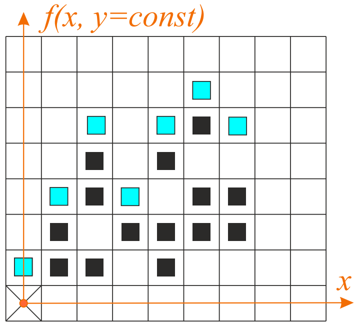


Sets to a function

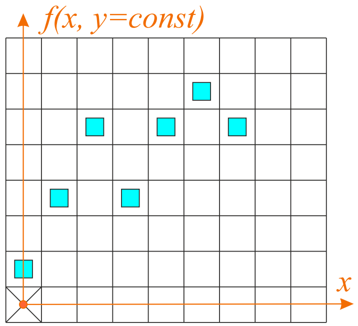
Top of the surface



1D function example



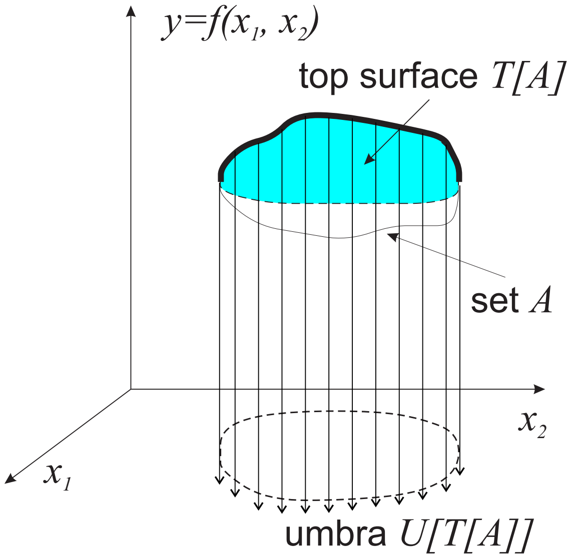
An arbitrary set



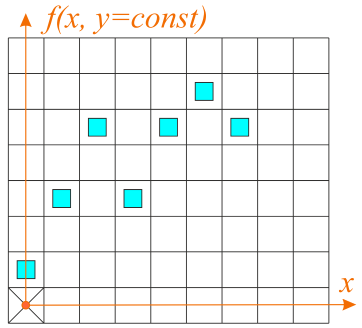
Top surface

- Let $A \subseteq \mathbb{E}^n$ is a domain $F = \{x \in \mathbb{E}^{n-1}$ for some $y \in \mathbb{E}$, $(x, y) \in A\}$.
- The top surface, top of the set A, is denoted $T[A]$ is a mapping $F \to \mathbb{E}$ defined as
  $[A](x) = \max\{y,\ (x, y) \in A\}$.

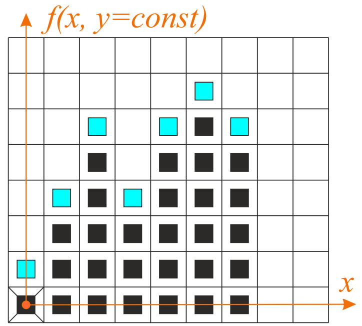
Umbra



1D function example



Top surface



Umbra

- Let $F \subseteq \mathbb{E}^{n-1}$ and $f\colon F \to \mathbb{E}$.
- The Umbra of a function (set) f, denoted $U[f]$, $U[f] \subseteq F \times \mathbb{E}$, $U[f] = \{(x, y) \in F \times \mathbb{E},\, y \leq f(x)\}$

Gray scale dilation, erosion by binary dilations, erosions

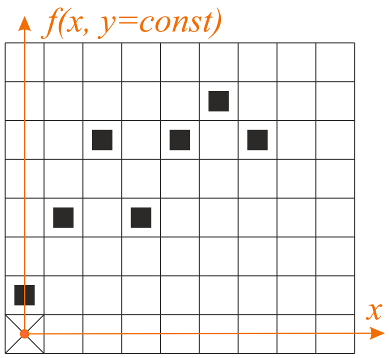
- The umbra function (set) U and the top surface function (set) T are used.
- Dilation:

$$f \oplus k = T[U[f] \oplus U[k]]$$

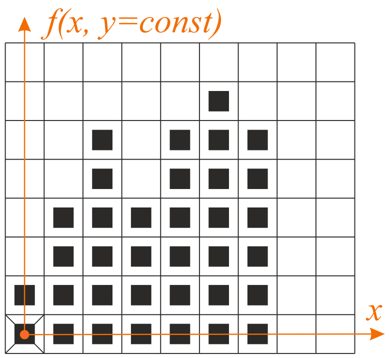
- Erosion:

$$f \ominus k = T[U[f] \ominus U[k]]$$

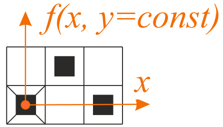
Grayscale dilation, 1D example



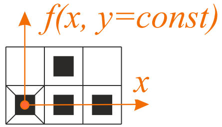
f



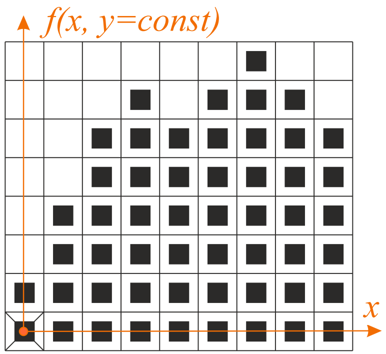
$U[f]$



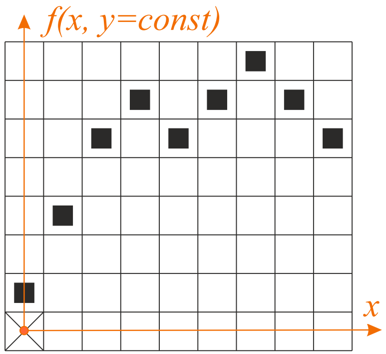
k



$U[k]$

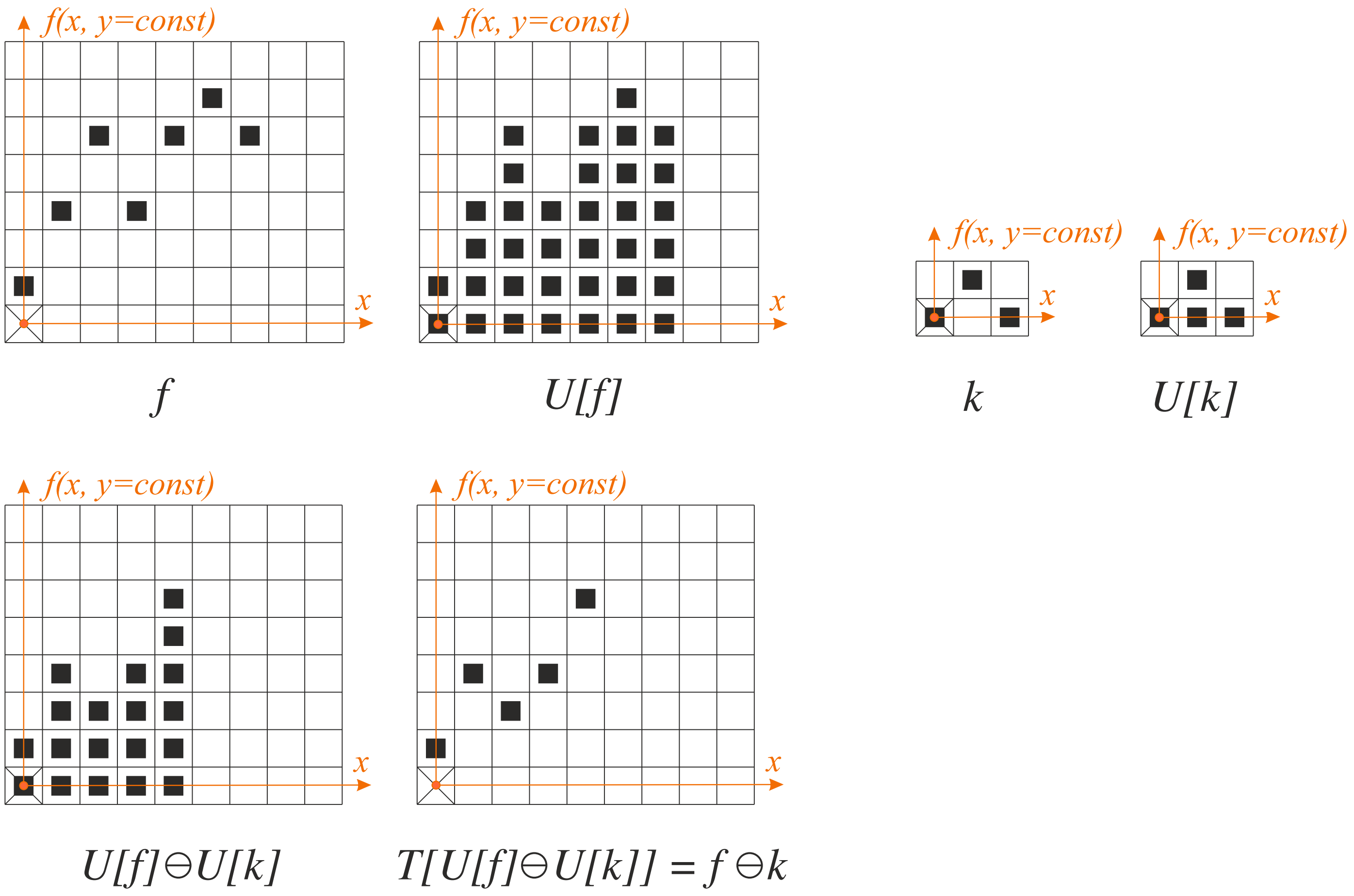


$U[f] \oplus U[k]$



$T[U[f] \oplus U[k]] = f \oplus k$

Grayscale erosion, 1D example



Grayscale dilation/erosion via lattice

- This is the alternative approach which uses the order structure in T in the lattice of functions T^E.
- The function g represents a structuring element.
- Dilation $(f \oplus g)(x) = \sup_{y \in Y} \{f(y) + g(x-y)\}$
- Erosion $(f \ominus g)(x) = \inf_{y \in Y} \{f(y) - g(x-y)\}$

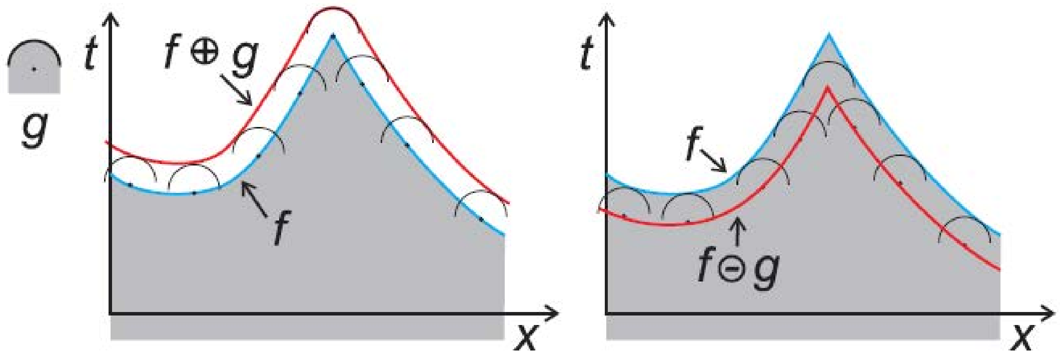


Image courtesy Petr Matula

Dilation/erosion with a flat structuring element

- Flat structuring elements g are defined to be equal to zero on a compact set K and to the value $\max(T)$ elsewhere
- We can write

$$f \oplus g = \sup_{y \in E,\, x-y \in K} f(y) = \sup_{y \in K_x} f(y)$$

$$f \ominus g = \inf_{y \in E,\, x-y \in K} f(y) = \inf_{y \in \breve{K}_x} f(y)$$

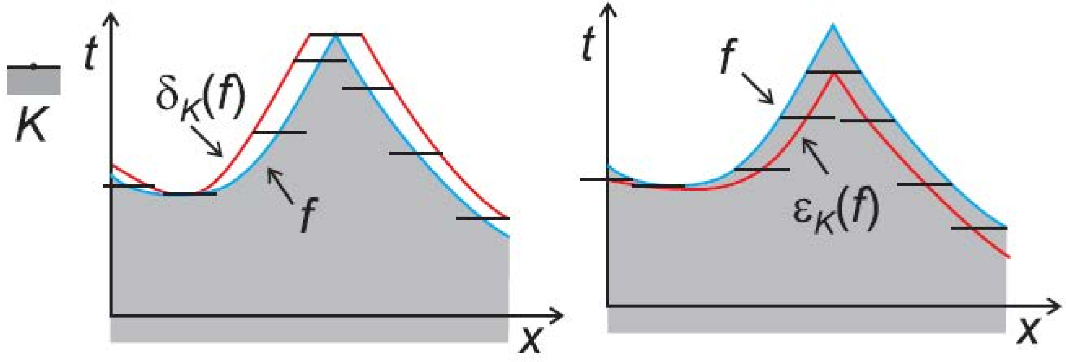


Image courtesy Petr Matula

Remarks, flat structuring element

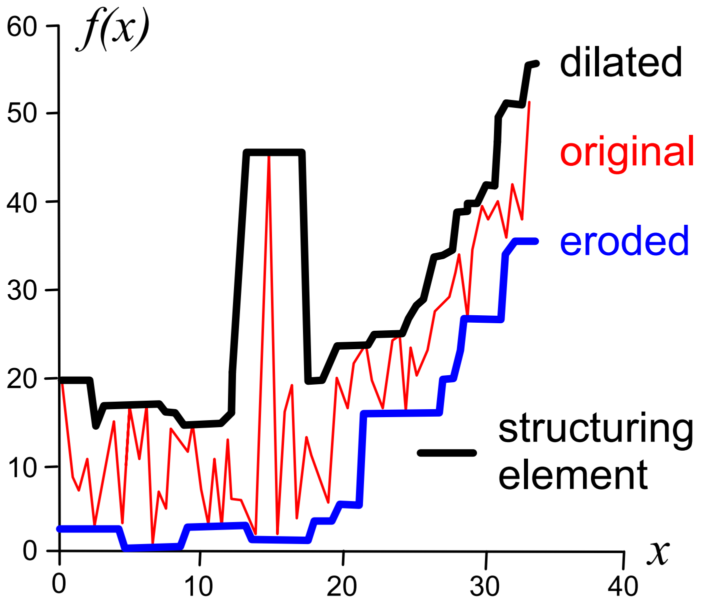
- Erosion

$$(f \ominus g)(x) = \inf_{y \in E,\, x-y \in K} f(y) = \inf_{y \in \breve{K}_x} f(y)$$

- Positive peaks are shrunk. Valleys are expanded.
- Dilation provides the dual effect.

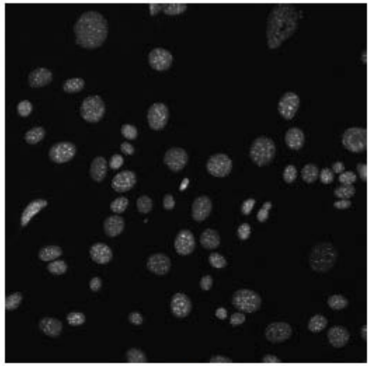
$$(f \oplus g)(x) = \sup_{y \in E,\, x-y \in K}$$

$$f(y) = \sup_{y \in K_x} f(y)$$



Courtesy J. Serra for the image idea.

Example: dilation, erosion with the flat structuring element



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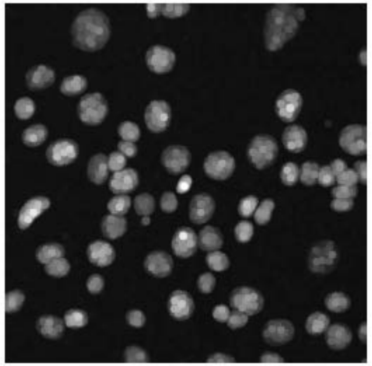
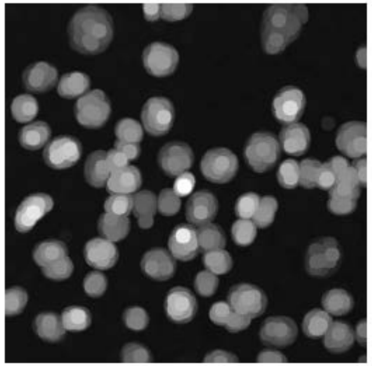
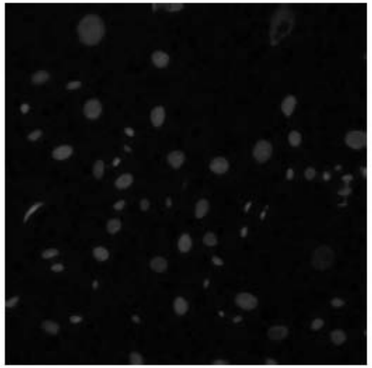


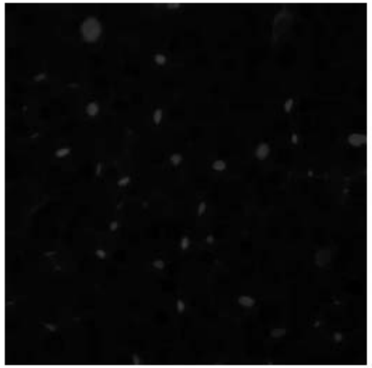
Dilation disk ϕ 10



Dilation disk ϕ 20



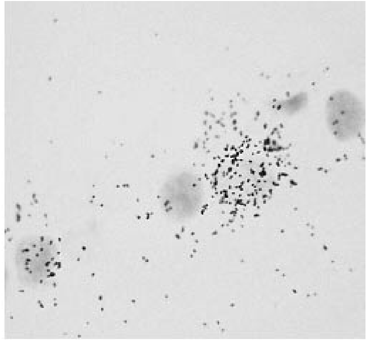
Erosion disk ϕ 10



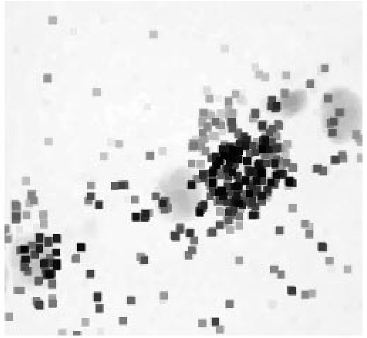
Erosion disk ϕ 20

Image courtesy Petr Matula

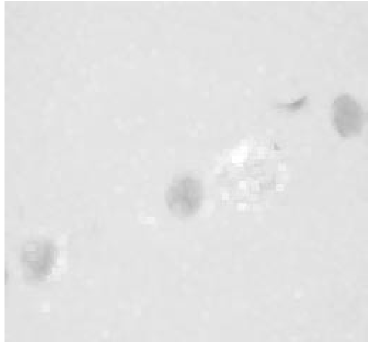
Example: grayscale morphological preprocessing



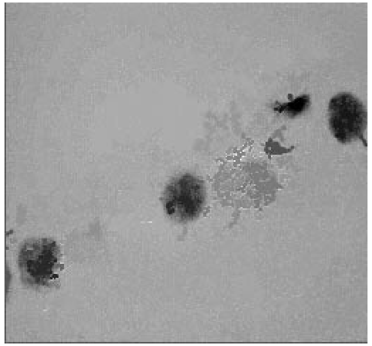
(a) original



(b) eroding dark



(c) dilating dark in (b)



(d) reconstr. cells

Remarks on grayscale dilation/erosion

- Dilations and erosions with a flat structuring element on grayscale images are equivalent to applying $\max$ and $\min$ filters.
- It is recommended to work on grayscale images as long as possible and defer thresholding at later times.
- Dilation/erosion compared with convolution

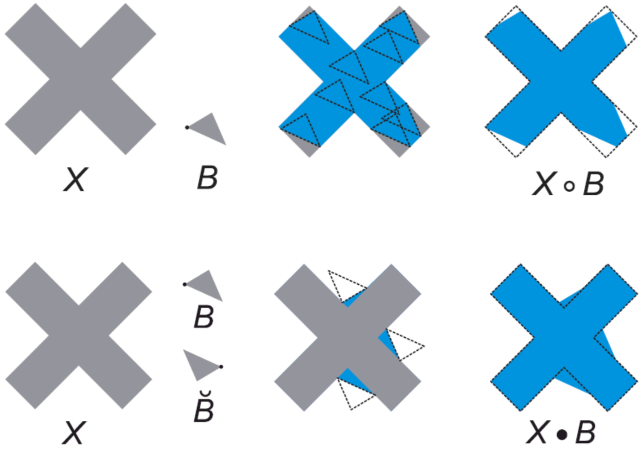
Convolution: $(f * g)(x) = \sum_{y \in Y} f(x-y) \cdot g(y)$

Dilation: $(f \oplus g)(x) = \sup_{y \in Y} \{f(y) + g(x-y)\}$

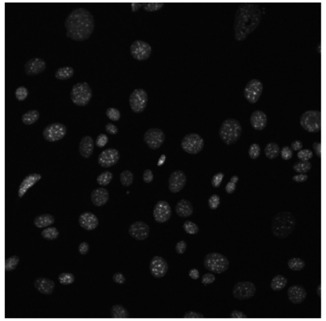
Convolution		*Dilation/erosion*	*Remark*
Summation	$\leftrightarrow$	$\sup$ or $\inf$	nonlinar
Product	$\leftrightarrow$	Summation	linear

Opening, closing

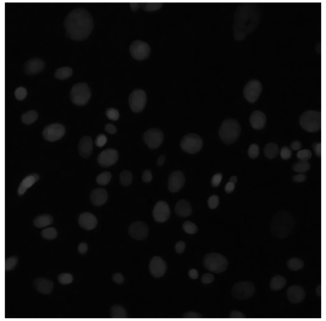
- Recall from the binary morphology lecture that a filter is a morphological filter if and only if it is increasing and idempotent.
- Grayscale dilation and erosion are morphological filters.
- Opening: $\gamma_B(X) = X \circ B = (X \ominus B) \oplus B$
- Closing: $\phi_B(X) = X \bullet B = (X \oplus B) \ominus B$



Opening and closing examples

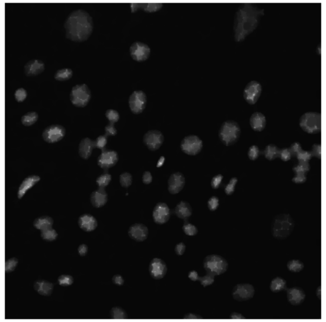


f

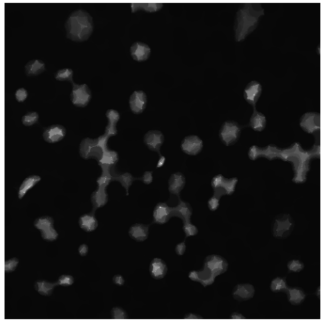


$\gamma_{Disk\varnothing 10}(f)$

$\gamma_{Disk\varnothing 20}(f)$



$\psi_{Disk\varnothing 10}(f)$



$\psi_{Disk\varnothing 20}(f)$

Image courtesy Petr Matula

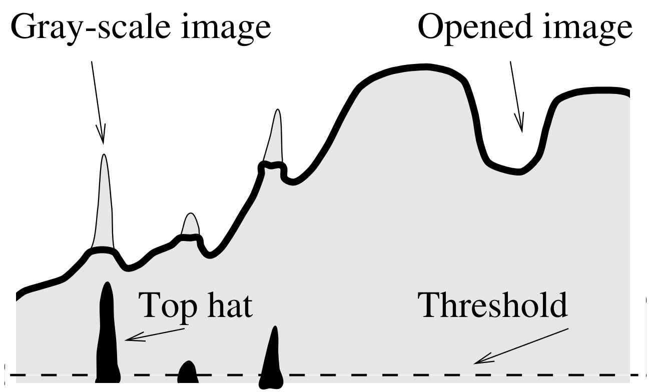
Grayscale hit or miss operation

- Dilations and erosion are more powerful when combined.
- E.g., introducing grayscale hit or miss operation which serves for template matching. Two structuring elements with a common representative point (origin). The first structuring element is the foreground pattern B_{fg}, the second one is the background pattern B_{bg}.
- Grayscale hit or miss operator is defined as

$$X \otimes B = (X \ominus B_{\mathrm{fg}}) \cap (X \ominus B_{\mathrm{bg}})$$

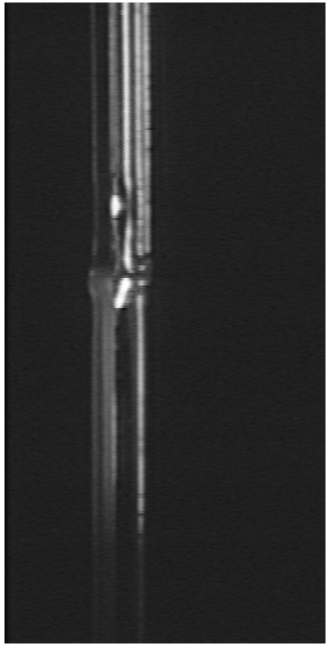
Top hat tranform

- Definition: $X \setminus (X \circ K)$.
- It is used for intensity-based object segmentation in the situation, in which the background intensity changes slowly.
- Parts of image larger than the structuring element K are removed. Only removed parts remain after subtraction, which are objects on the more uniform background now. The objects can be found by thresholding now.

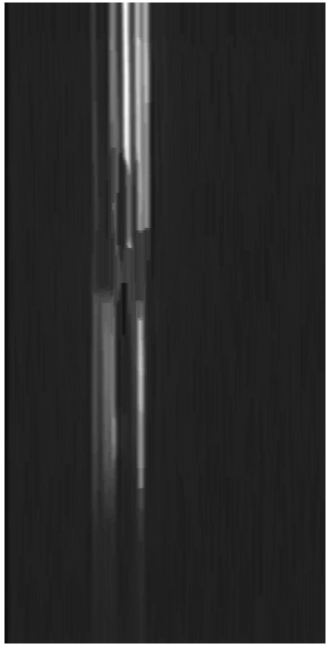


Example: Production of glass capillaries for thermometers

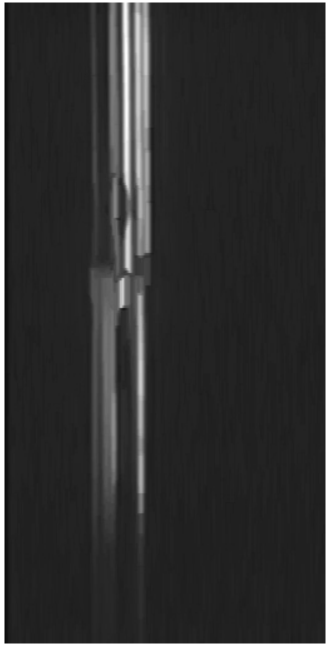
Top-hat transform illustrated on the industrial example.



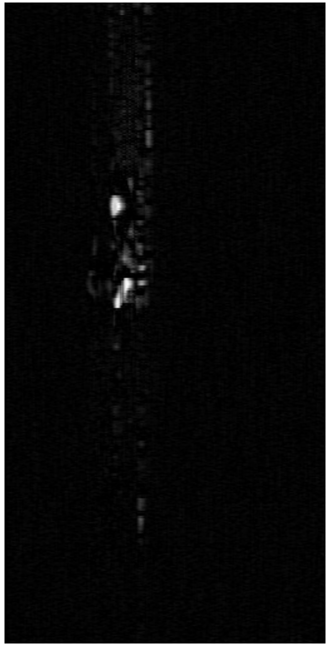
Original
512×256



Erosion with struct.
elem. 1×20



Opening with the
same struct. elem.



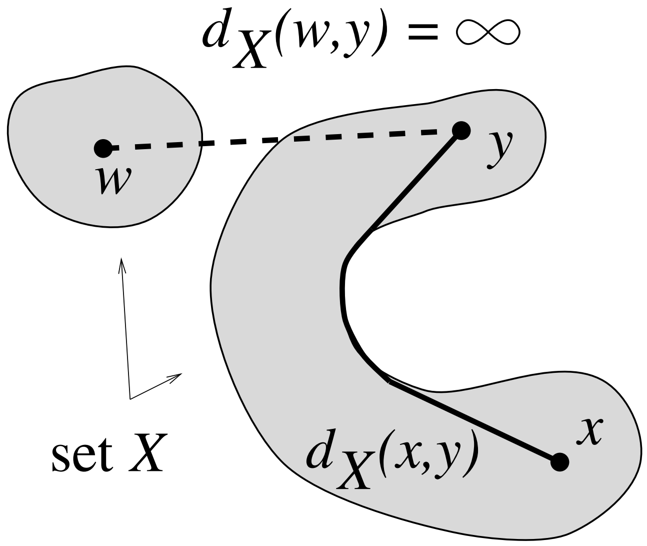
Resulting
segmentation

Geodesic method in mathematical morphology

- A geodesic method change morphological operations in such a manner that they operate on the part of the object only.
- Geodesic methods offer a unifying framework describing the local geometry of images and surfaces. Fast and efficient algorithms compute geodesic distances to a set of points and shortest paths between points.
- Example: Assume that we have reconstruct the object from the marker, say a cell from the cell nucleus. In such a case, it is desirable to prevent the growth outside of the cell.
- We will see later that the structuring element can change in every pixel based on image function values in a local neighborhood.

Geodesic distance

- Geodesic distance $d_X(x, y)$ is the length of the shortest path between two points x, y under the condition that they belong to the set X.
- If there is no path connecting x, y then the geodesic distance is defined as $d_X(x, y) = +\infty$.

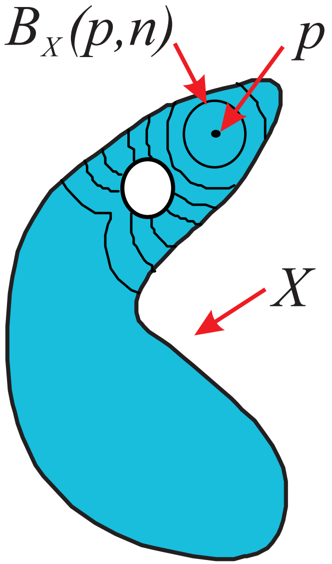


Geodesic circle (ball, hyperball)

- Geodesic circle (ball, hyperball for the space dimension > 3) is a circle (ball, hyperball) constrained to set X.
- Geodesic circle $B_X(p, n)$ with the center $p \in X$ and the radius n is defined

$$B_X(p, n) = \{p' \in X,\ d_X(p, p') \leq n\} \,.$$

- We can use dilation/erosion only inside the the subset Y of the image X.



Conditional dilation

- Serves as the basis of geodetic transformations and morphological reconstruction.
- Conditional dilation of a set X (called marker) by a structuring element B using a reference set R (called mask)

$$\delta_{R,B}^{(1)} = (X \oplus B) \cap R \,,$$

  where the superscript $^{(n)}$ gives the size of the dilation, in this special case $n = 1$.
- It is obvious that $\delta_{R,B}^{(1)} \subseteq R$.
- The set B is usually small (often a basic structuring element induced by the underlying grid). Set B is often omitted in subscripts.

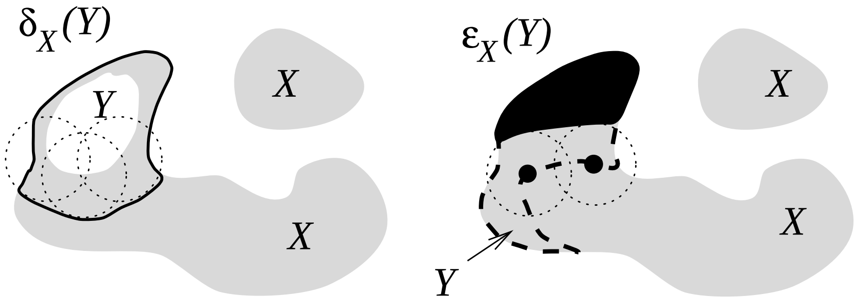
Conditional geodesic dilation and erosion

- Geodesic dilation $\delta_X^{(n)}(Y)$ of size n of a set Y inside the set X,

$$\delta_X^{(n)}(Y) = \bigcup_{p \in Y} B_X(p, n) = \left\{p' \in X,\ \exists p \in Y,\ d_X(p, p') \leq n\right\}.$$

- Geodesic erosion $\epsilon_X^{(n)}(Y)$ of size n of a set Y inside the set X,

$$\epsilon_X^{(n)}(Y) = \left\{p \in Y,\ B_X(p, n) \subseteq Y\right\} = \left\{p \in Y,\ \forall p' \in X \setminus Y,\ d_X(p, p') > n\right\}.$$



Geodesic dilation, erosion, implementation

- The simplest geodesic dilation of size one ($\delta_X^{(1)}$) of a set Y (marker) inside X is obtained as the intersection of the unit-size dilation of Y (with respect to the unit ball B) with the set X

$$\delta_X^{(1)} = (Y \oplus B) \cap X \; .$$

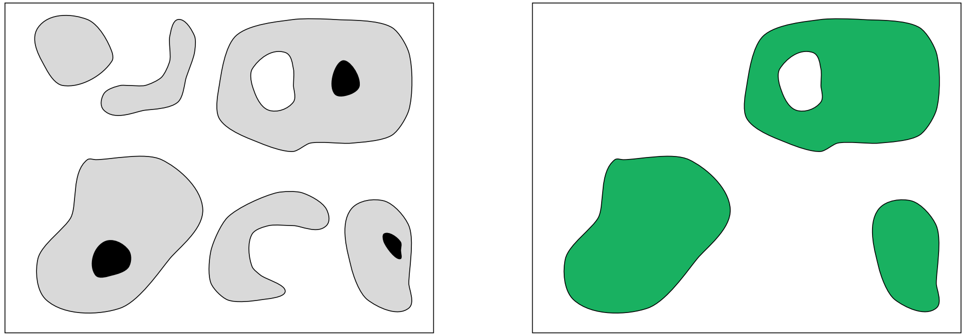
- Larger geodesic dilations are obtained by iteratively composing unit dilations n times

$$\delta_X^{(n)} = \underbrace{\delta_X^{(1)}\Big(\delta_X^{(1)}\big(\delta_X^{(1)} \dots (\delta_X^{(1)})\big)\Big)}_{n \text{ times}} \; .$$

- The fast iterative way to calculate geodesic erosion is similar.

Morphological reconstruction, motivation

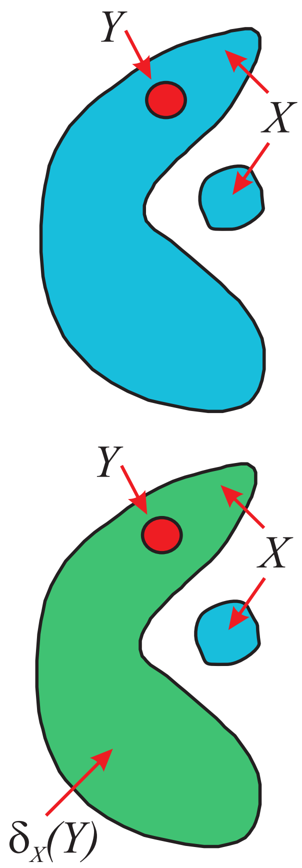
- Assume that we want to reconstruct objects of a given shape from a binary image that was originally obtained by thresholding segmentation. The set X is a union of all connected components of all thresholding results.
- However, only some of the connected components were marked either manually or automatically by markers that represent the set Y.
- The task is to reconstruct marked regions



Reconstruction of X (shown in light gray) from markers Y (black). The reconstructed result is shown in green on the right side.

Morphological reconstruction

- Successive geodesic dilations of the set Y inside the set X reconstruct the connected components of X marked initially by Y.
- When dilating from markers Y, connected components of X not containing Y disappear.
- Geodesic dilations terminate when all connected components set X previously marked by Y are reconstructed, i.e., idempotency is reached, i.e. $\forall n > n_0$, $\delta_X^{(n)}(Y) = \delta_X^{(n_0)}(Y)$.
- This operation is called reconstruction and denoted by $\rho_X(Y)$. Formally $\rho_X(Y) = \lim_{n\to\infty} \delta_X^{(n)}(Y)$.
- Reconstruction by dilation is an opening w.r.t. Y and closing w.r.t. X.

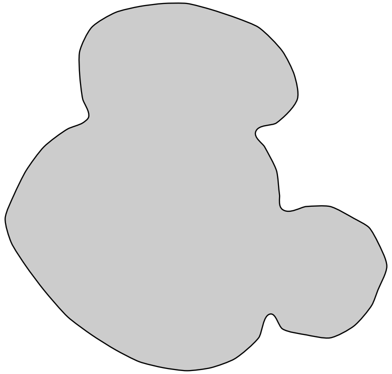


Automatic object marking, the idea

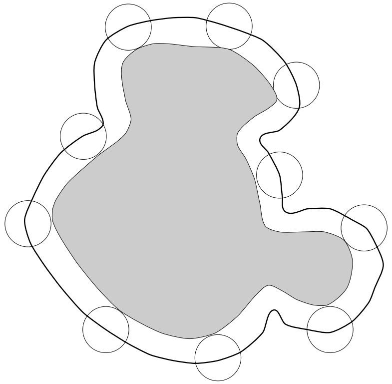
- The idea: Consider the convex region in the binary image and its shape only. The region can be represented by the marker 'inside the region'.
- It holds for non-touching circles trivially.
- The situation is more complicated in general.
- The sequential erosion is used. The residual region, i.e. the region which disappears at last while sequentially eroding is used as the marker. This motivates the ultimate erosion concept.
- Nonconvex regions are usually divided into simpler convex parts.

- The explanation plan:
  - Quench function – associates each point of the skeleton to a radius of an inscribed circle.
  - Several types of extremes in digitized functions (images).
  - Ultimate erosion.

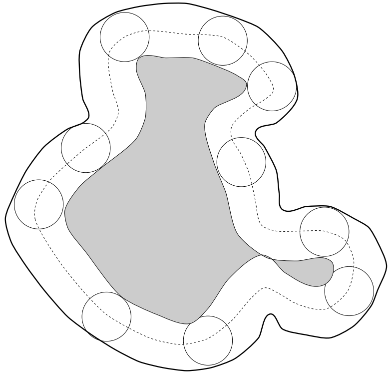
Sequential eroding, the example



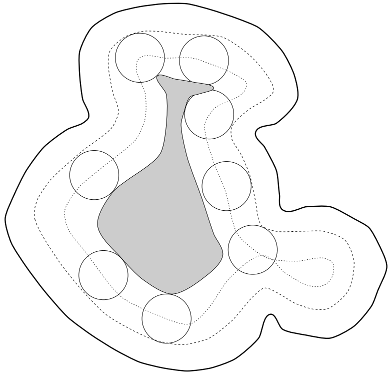
Original



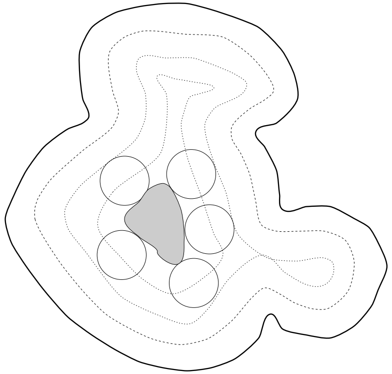
1st erosion



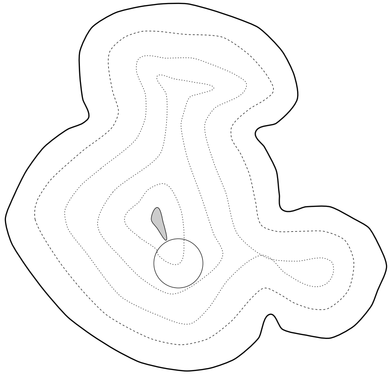
2nd erosion



3rd erosion



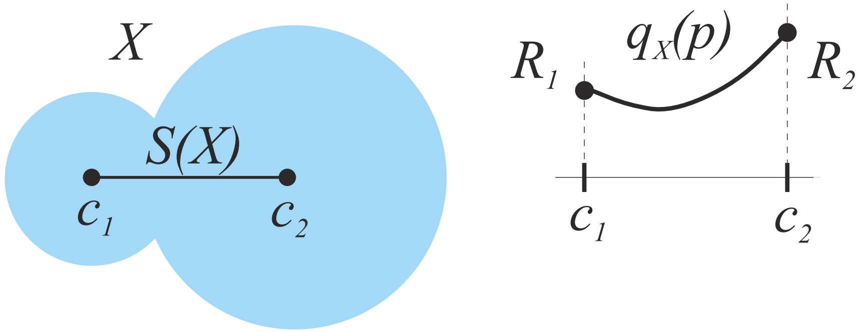
4th erosion



5th erosion

Quench function

- The binary point set (a 2D region) X can be equivalently representeed using maximal balls B.
- Every point p of the skeleton $S(X)$ by maximal balls has an associated ball B of radius $q_X(p)$.
- The term quench function is used for this association.
- Example: Quench function for two overlapping discs.
  c_1, c_2 are centers of discs. R_1. R_2 are respective disc radii.
  The quench function $q_X(p)$ is on the right side of the figure.



  Skeleton $S(X)$ of the binary image X consisting of two overlapping discs.
- *Later, analyzing various types of quench function maxima will be used in the ultimate erosion definition.*

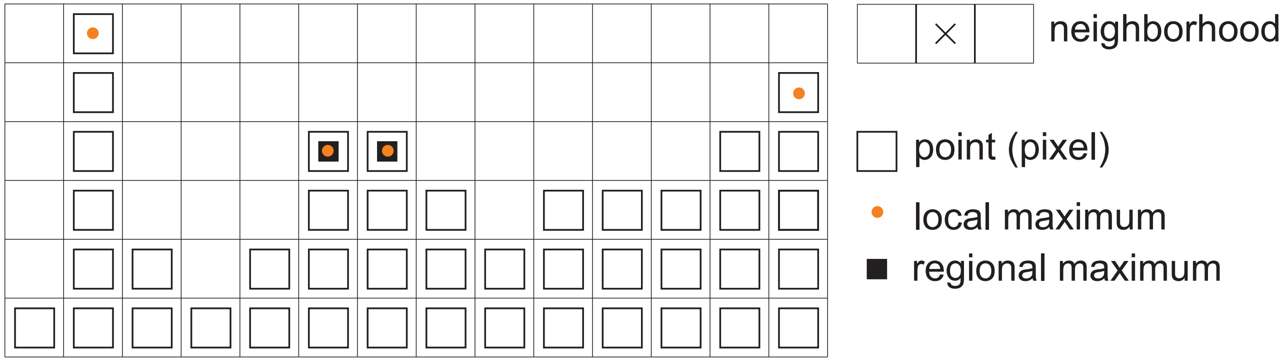
A region as a union of maximal balls

- Recall from previous slide that every point p of the skeleton $S(X)$ by maximal balls has an associated ball B of radius $q_X(p)$.
- If the quench function $q_X(p)$ is known for each point of the skeleton then the original underlying point set (a 2D region) X can be reconstructed as the union of maximal balls B

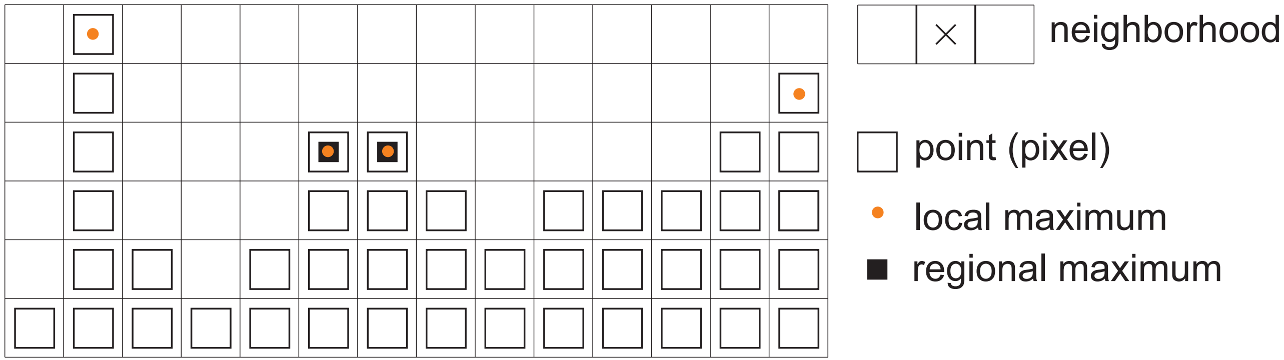
$$X = \bigcup_{p \in S(X)} \left(p + q_X(p) B \right) .$$

Three types of extremes of the grayscale image function I

- The global maximum of the image (also image function) $I(p)$ is represented by the pixel (pixels) p having the highest value of $I(p)$ (analogy to the highest point in the landscape).
- The local maximum is pixel p iff it holds for each neighboring pixel q of the pixel p that $I(p) \geq I(q)$.
- The regional maximum M of the image $I(p)$ is a contiguous set of pixels with the image function value h (landscape analogy: plateau at the altitude h), where each pixel neighboring to the set M has a lower value than h.



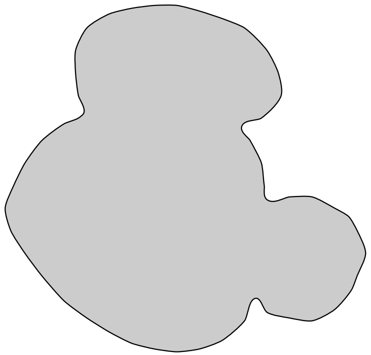
Not all local maxima are regional maxima



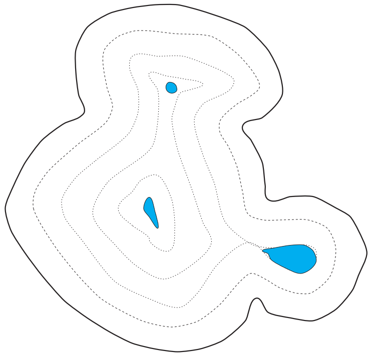
- Each pixel of the regional maximum M of the image function I is also the local maximum.
- The contrary does not hold, i.e. there are local maxima, which are not regional maxima.

Ultimate erosion $Ult(X)$

- The ultimate erosion outcome is often used as automatically created markers of convex objects in binary images.
- The situation becomes more complicated when convex regions overlap, which may induce non-convexity. Recall two overlapping circles example in slide 31.
- Ultimate erosion $Ult(X)$ is the set consisting of quench function $q_X(p)$ regional maxima.
- Example: Ultimate erosion as a union of connected component residuals before they disappear while eroding.



Original binary image



Ultimate erosion outcome

Ultimate erosion expressed as the reconstruction

- $\mathbb{N}$ is the set of natural numbers, which will serve us to characterize growing circle radii.
- Ultimate erosion can be expressed as

$$Ult(X) = \bigcup_{n \in \mathbb{N}} \left((X \ominus nB) \setminus \rho_{X \ominus nB}(X \ominus (n+1)B) \right) \; .$$

- An effective calculation of the ultimate erosion relies on the distance transform algorithm which was explained in the lecture Digital image.

Fast calculations using the distance tranformation

- Distance transformation (function) $dist_X(p)$ assigns to each pixel p from the set X the size of the first erosion of a set, which does not contain the pixel p, i.e.

$$\forall p \in X, \quad dist_X(p) = \min \{ n \in \mathbb{N} \, , \, p \text{ not in } (X \ominus nB) \} \, .$$

- $dist_X(p)$ is the shortest distance between the pixel p and the set complement X^C.

The distance function has two direct uses:

- The ultimate erosion of a set X is constituted by a union of regional maxima of the distance function of the set X.
- The skeleton created by maximal circles of the set X is given by the set of local maxima of the distance function X.

Skeleton by influence zones (SKIZ)

- Let X be composed of n connected components X_i, $i = 1, \dots, n$.
- The influence zone $Z(X_i)$ consists of points which are closer to set X_i than to any other connected component of X.

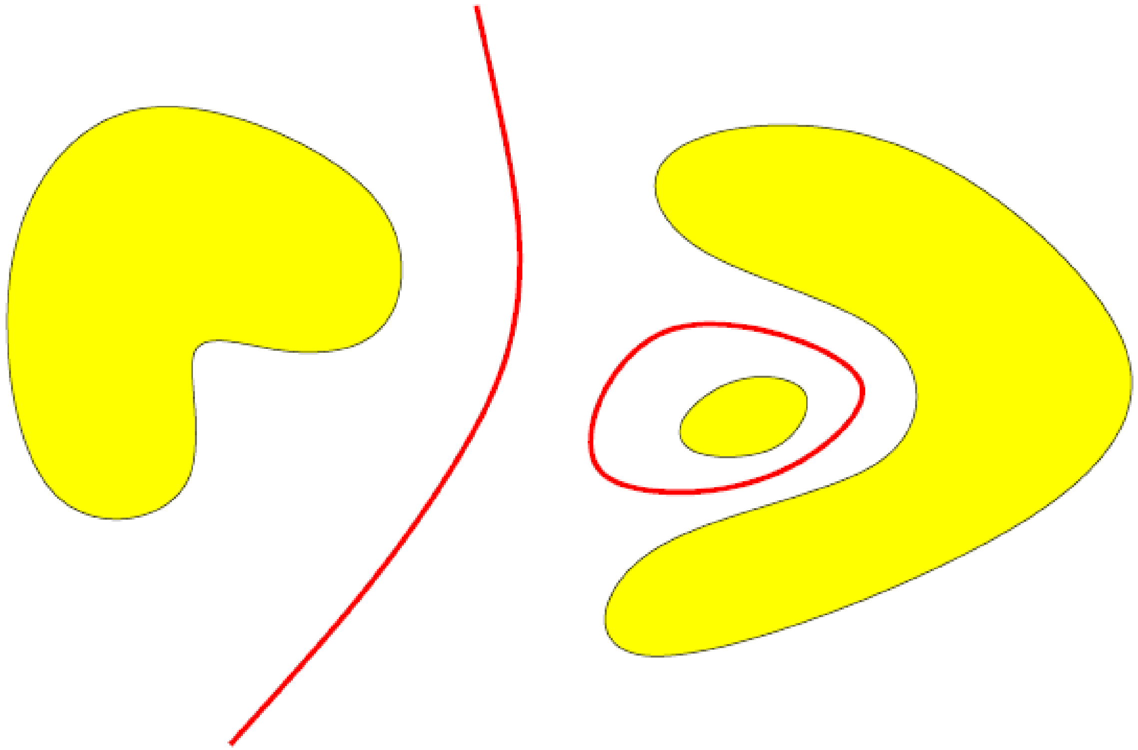
$$Z(X_i) = \left\{ p \in \mathbb{Z}^2,\ \forall i \neq j,\ d(p, X_i) \leq d(p, X_j) \right\} .$$

- The skeleton by influence zones denoted $\mathrm{SKIZ}(X)$ is the set of boundaries of influence zones $\{Z(X_i)\}$.

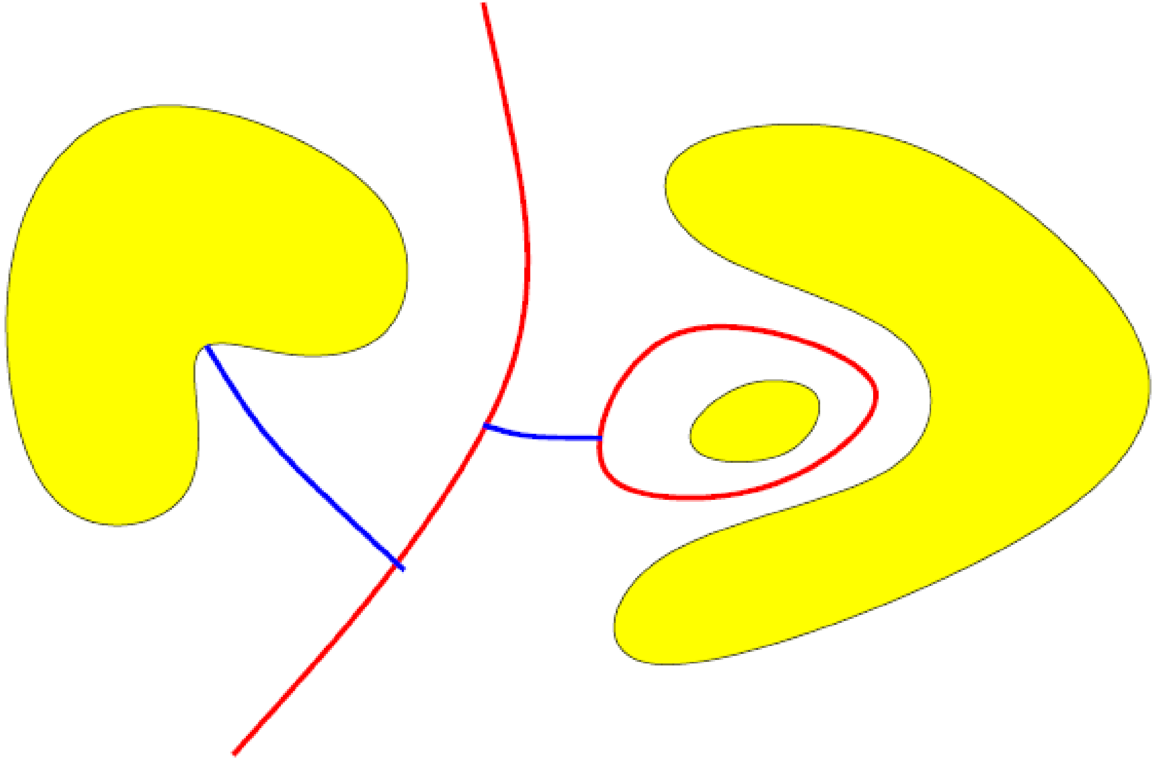
$$\mathrm{SKIZ}(X) = \left(\bigcup_i Z(X_i) \right)^C .$$

- Properties:
  - $\mathrm{SKIZ}(X)$ is not necessarily connected (even if X^C is).
  - $\mathrm{SKIZ}(X) \subseteq \mathrm{Skeleton}(X)$.

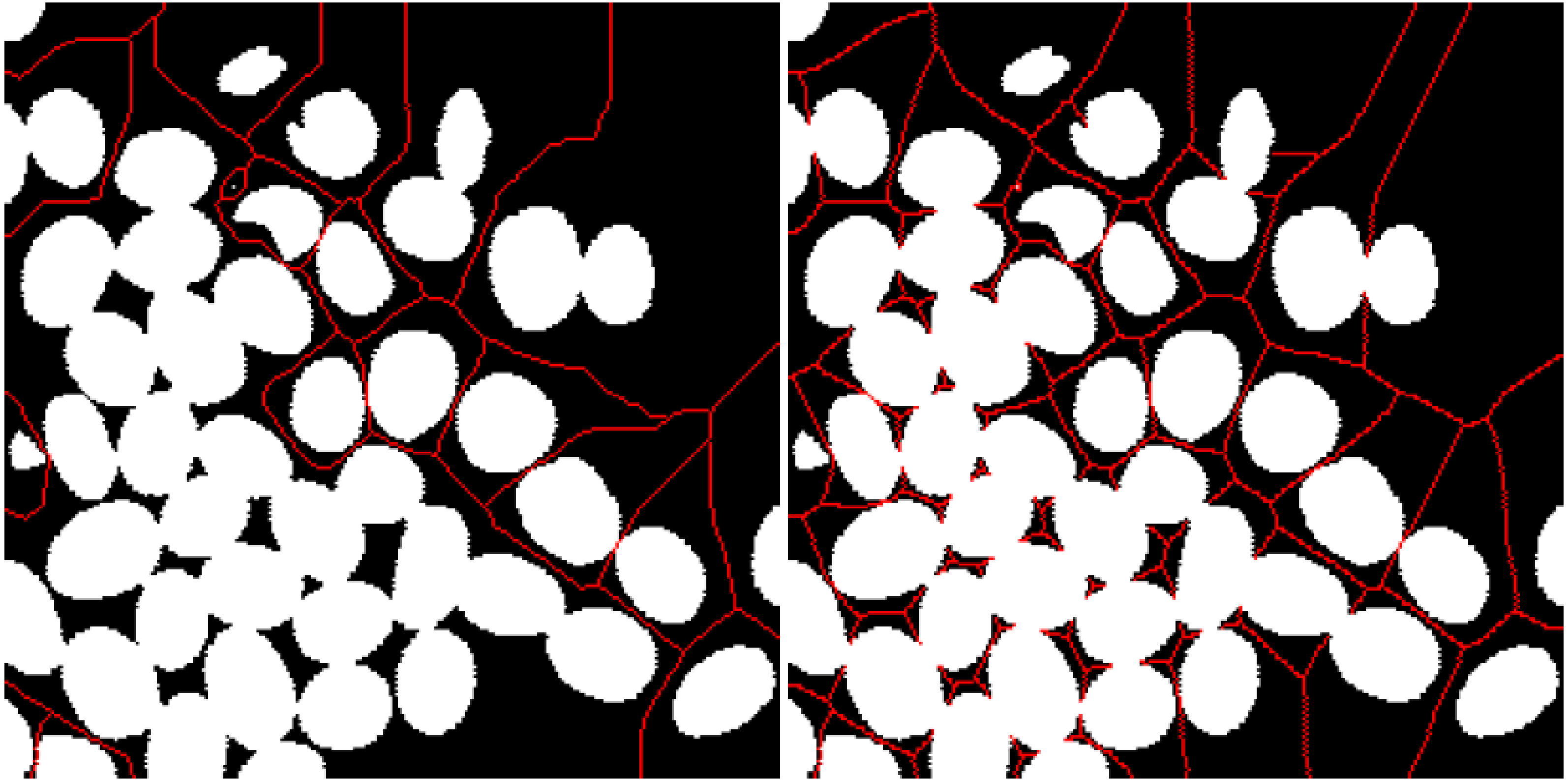
SKIZ idea, example



SKIZ(X) ⊆ Skeleton(X)



SKIZ$(X) \subseteq$ Skeleton(X), particles example



Several markers for a region, issues
Geodesic influence zone

- In some applications, it is desirable that one connected component of X is marked by several markers Y.
- If the above is not acceptable then the notion of influence zones can be generalized to geodesic influence zones of the connected components of set Y inside X.