SMT and Functional Equation Solving over the Reals: Challenges from the IMO

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CADE, 28 July 2025

International Mathematical Olympiad

- Competition for pre-university students.
- Many other smaller (regional) events.
- Range of problems.
- LLMs are going at it (as we speak).
- How do SMT/ATP fair?

Find All Functions : “All-Synth”

Problem:

Given a specification for a function f, find all such f.

Where:

The class of solutions has a reasonable description.

In the competition participants are also expected to provide a proof.

Example

Input

$$\forall xy : \mathbb{R}.\ f(x+y) = xf(y) + yf(x)$$

$$\forall x : \mathbb{R}.\ f(x) = xf(0) \quad \text{with } y \mapsto 0$$

$$f(0) = 0 \quad \text{with } x, y \mapsto 0$$

Solution

$$f = \lambda x.\ 0$$

What Is Reasonable? Examples

Spec: $\forall x, y : f(x) + f(y) = f(f(x)f(y))$
Sol: $\lambda x.\, 0$

Spec: $\forall x, y : f(x + y) + f(x)f(y) = f(xy) + f(x) + f(y)$
Sol: $\lambda x.\ x, \lambda x.\ 0, \lambda x.\ 2$

Spec: $\forall x, y : \mathbb{R}.\, f(x + y) = f(x) + y$
Sol: $\lambda x.\ x + c,\ \ c \in \mathbb{R}$

Template-and-QE

[Brown et al. SC^2 2024]

1. Guess a template for f, e.g. linear
2. Inline template.
3. Perform Quantifier Elimination over $\forall x$
4. Prove there are no more solutions (outside the template).

Template-and-QE, Example

Inline linear template

$$\forall xy.\, f(x+y) = xf(y) + yf(x)$$
$$\forall xy.\, a(x+y) + b = x(ay+b) + y(ax+b)$$

QE $\forall xy$

$$a = b = 0$$

Prove no more solutions

$$\forall xy.\, f(x+y) = xf(y) + yf(x) \Rightarrow \forall x.\, f(x) = 0$$

Proving No-More-Solutions Bottleneck

Negate+Skolemize

$$\forall xy.\, f(x+y) = xf(y) + yf(x) \wedge f(c) \neq 0$$

Instantiate

$f(0) = 0$ $\quad x \mapsto 0, y \mapsto 0$
$f(c) = cf(0)$ $\quad x \mapsto c, y \mapsto 0$
Contradiction

Partial Instantiations with Simple Terms

- CVC5 has enumerative mode. [Janota et al. FMCAD '21]
- Enumeration only takes into account existing ground terms and does not have domain knowledge.
- $0, 1, c$ are often useful
  — not always present in the formula.
- Make the solve focus on instantiations with 0/1.

Theory-Unification by Equation Solving

Example

$\forall xy.\ f(x+y) - f(x-y) = xy$

Nice instantiation

$\forall z.\ f(z) - f(0) = z^2/4$ $\quad x \mapsto z/2, y \mapsto z/2$

Look for a substitution that zeroes all arguments except for one, where it's just z.

Lemma Generation

1. Generate candidate lemmas,
   e.g. $f(0) = 0$, $f(1) = 0$, $\ldots$
2. Try to prove lemmas.
3. If proven, add to the original problem.

- Lemmas by brute-force up to certain depth.
- Lemmas from target solution,
  E.g. if proving for $f(x) = x \lor f(x) = -x$,
  generate candidate $f(c) = c \lor f(c) = -c$.

Results

- Hand-formalized data from a document by Musil.
- Scraped data + LLM to formalize from Art of Problem Solving.
- TARSKI for QE
- VAMPIRE, CVC5, Z3 in portfolio for SMT

		Def.	−EQ	−EQ+FI	−PI	−TU	−PI−TU	−L	−PI−L	−TU−L	Base	VBS
Musil	tot.	21	21	20	18	17	13	19	17	14	13	22
	uniq	0	0	1	0	0	0	0	0	0	0	
AoPS	tot.	77	75	64	60	76	47	64	39	58	33	87
	uniq	2	1	5	2	1	0	0	0	0	0	

Summary

- Targeting math competition problems "all-synth".
- First attempt: template-and-QE
- Proving that we have all the solutions is hard.

Helping SMT

- Simple partial instantiations.
- Theory reasoning for equalities.
- Adding lemmas.