Challenges and Solutions for Higher-Order SMT Proofs

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Motivation

- Proofs for SMT are a long-standing challenge

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- Semantics is becoming more of a challenge (SMT3)

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This is, however, no accident. Because of the ever increasing number of logical theories supported by SMT solvers, the variety of deductive systems used to describe the various solving algorithms, and the relatively young age of the SMT field, designing a single set of axioms and inference rules that would be a good target for all solvers does not appear to be practically feasible."

— Stump et al., Formal Methods in System Design 2013

Plan

SMT proofs

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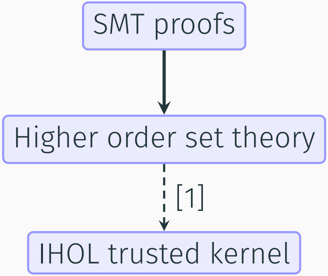
SMT proofs

↓

Higher order set theory

Plan



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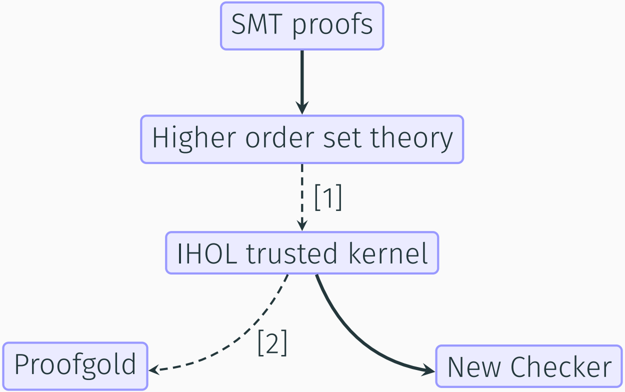

SMT proofs
Higher order set theory
[1]
IHOL trusted kernel
[2]
Proofgold
New Checker

Flavors of Translation to Set Theory

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p is true	$0 \in p$

(Trusted) Kernel

- intuitionistic higher logic as the underlying trusted kernel

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(Trusted) Kernel

- intuitionistic higher logic as the underlying trusted kernel
- each proof gives a proof term

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- Prove UNSAT: $\forall p \in 2. (0 \in p \rightarrow 0 \notin p \rightarrow \bot)$
- Remarks: $\rightarrow$ is IHOL built-in, $\in$ is axiomatized
- Proof term: $\lambda p : \iota.\lambda u : p \in 2.\lambda v : 0 \in p.\lambda w : 0 \notin p.w\ v$

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  - there is no array h bijective from `Int` to `Int`

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- Are we happy about this result?

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- Example issue: Type-checking of parametric bitvectors
  Type-checks? $\texttt{bv}_1[n]\ \texttt{++}\ \texttt{bv}_2[m] = \texttt{bv}_2[m]\ \texttt{++}\ \texttt{bv}_1[n]$

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- Heavier kernel

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- Proofgold checker: part of Proofgold
- New checker: can be used independently

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- Proposal:
  Higher order set theory
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  obtaining small trusted kernel
- Natural translation of SMT concepts to sets.
- Feasibility on concrete examples and available checkers.

Chad E. Brown and Karol Pąk.
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